

Universality and critical behavior in the chiral two-matrix model

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Abstract

We study the chiral two-matrix model with polynomial potential functions V and W , which was introduced by Akemann, Damgaard, Osborn and Splittorff. We show that the squared singular values of each of the individual matrices in this model form a determinantal point process with correlation kernel determined by a matrix-valued Riemann-Hilbert problem. The size of the Riemann-Hilbert matrix depends on the degree of the potential function W (or V respectively). In this way we obtain the chiral analogue of a result of Kuijlaars-McLaughlin for the non-chiral two-matrix model. The Gaussian case corresponds to V, W being linear.

For the case where $W(y) = y^2/2 + \alpha y$ is quadratic, we derive the large n -asymptotics of the Riemann-Hilbert problem by means of the Deift-Zhou steepest descent method. This proves universality in this case. An important ingredient in the analysis is a third-order differential equation.

Finally we show that if also $V(x) = x$ is linear, then a multi-critical limit of the kernel exists which is described by a 4×4 matrix-valued Riemann-Hilbert problem associated to the Painlevé II equation $q''(x) = xq(x) + 2q^3(x) - \nu - 1/2$. In this way we obtain the chiral analogue of a recent result by Duits and the second author.

Keywords: chiral random matrix theory, two-matrix model, Riemann-Hilbert problem, universality, multi-criticality, singular values, determinantal point process.

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1 Introduction

Chiral random matrix theory provides a useful calculational framework for solving various physical problems, especially in quantum chromodynamics. Originally this was developed for the chiral 1-matrix model with a Gaussian potential [42, 44, 45].

Recently a chiral 2-matrix model was introduced by Akemann-Damgaard-Osborn-Splittorff [4]. The model is a Hermitian analogue of a non-Hermitian model introduced earlier by Osborn [40]. The motivation for this matrix model comes again from quantum chromodynamics. It turns out that the mixed correlation functions for $n \rightarrow \infty$ strongly depend on a certain constant in the model, which has a physical interpretation as the ‘pion decay constant’ F_π . This suggests a way to compute F_π via numerical lattice simulations. A numerical implementation of this idea is provided in [14] and some recent developments are, e.g., in [36]. The method for calculating F_π was originally developed without random matrices [12, 13]. The chiral two-matrix model yields an elegant theoretical framework that brings this method to its full power, see also [4, 7] for the connection between the matrix model and the limiting physical theory.

The chiral two-matrix model in [4] in its simplest, i.e. ‘quenched’ form, is defined by the probability distribution

$$\frac{1}{\tilde{Z}_n} \exp(-n \text{Tr}(\Phi^* \Phi + \Psi^* \Psi)) \, d\Phi \, d\Psi,$$

defined on pairs of rectangular complex matrices (Φ, Ψ) , both of size $n \times (n + \nu)$, where n and ν are integers, the superscript $*$ stands for the conjugate transpose, and \tilde{Z}_n is a normalization constant. Here $d\Phi$ and $d\Psi$ are the flat complex Lebesgue measures on the entries of Φ and Ψ . After the change of variables, see [4],

$$\Phi_1 = \Phi + \mu_1 \Psi, \quad \Phi_2 = \Phi + \mu_2 \Psi, \quad \mu_1, \mu_2 \in \mathbb{R},$$

this becomes

$$\frac{1}{Z_n} \exp(-n \text{Tr}(c_1 \Phi_1^* \Phi_1 + c_2 \Phi_2^* \Phi_2 - \tau(\Phi_1^* \Phi_2 + \Phi_2^* \Phi_1))) \, d\Phi_1 \, d\Phi_2, \quad (1.1)$$

for certain constants $c_1, c_2 > 0$ and $\tau \in \mathbb{R}$ and a normalization constant Z_n . Here c_1, c_2, τ are known functions of μ_1 and μ_2 . We assume without loss of generality that $\tau > 0$. For this measure to be finite we also need $0 < \tau^2 < c_1 c_2$.

As noted in [4], one can consider the more general model

$$\frac{1}{Z} \exp(-n \text{Tr}(V(\Phi_1^* \Phi_1) + W(\Phi_2^* \Phi_2) - \tau(\Phi_1^* \Phi_2 + \Phi_2^* \Phi_1))) \, d\Phi_1 \, d\Phi_2, \quad (1.2)$$

where V, W are polynomials with positive leading coefficient. This is the setting that we will consider in this paper. We assume without loss of generality that ν is a nonnegative integer.

We note that the general model in [4] has extra determinantal factors in the probability distributions (1.1)–(1.2). What we consider here is the ‘quenched’ case where there are no determinants of this kind. In fact, the correlation functions of the general model can be expressed in terms of their quenched variants [4, 6].

The chiral two-matrix model can be analyzed with the help of biorthogonal polynomials [4, 26]. The monic polynomials are denoted $P_{j,n}(x)$ and $Q_{k,n}(y)$, of degree j and k respectively, and they satisfy the biorthogonality relations

$$\int_0^\infty \int_0^\infty w_n(x, y) P_{j,n}(x) Q_{k,n}(y) \, dx \, dy = \kappa_k \delta_{j,k}, \quad \kappa_k \neq 0, \quad j, k = 0, 1, 2, \dots, \quad (1.3)$$

with respect to the weight function

$$w_n(x, y) = (xy)^{\nu/2} I_\nu(2\tau n \sqrt{xy}) e^{-n(V(x) + W(y))}, \quad (1.4)$$

see [4, Eqs. (2.15) and (2.16)], with I_ν the modified Bessel function

$$I_\nu(x) = \sum_{k=0}^\infty \frac{1}{k! \Gamma(k + \nu + 1)} (x/2)^{2k + \nu}. \quad (1.5)$$

The weight function is well-defined for all $\nu > -1$, not necessarily an integer. Following the approach by Ercolani-McLaughlin [25] one shows that the biorthogonal polynomials exist, are unique, and have real and simple zeros. In the case where V, W are both linear, the biorthogonal polynomials are explicitly known to be given by Laguerre polynomials [4].

Incidentally, there also exists a non-Hermitian version of the chiral two-matrix model, due to Osborn [40]. In that case, $I_\nu(x)$ in (1.4) is replaced by the modified Bessel function of the second kind $K_\nu(x)$, which is defined in (4.12) below.

Our interest lies in the singular values of the matrix Φ_1 . They form a determinantal point process with a correlation kernel called H_N in [4]. We find it more convenient to study the determinantal point process of the squared singular values of Φ_1 which is then described by the correlation kernel

$$K_n(x_1, x_2) = \sum_{k=0}^{n-1} \frac{1}{\kappa_k} \left(\int_0^\infty w_n(x_1, y) Q_{k,n}(y) \, dy \right) P_{k,n}(x_2), \quad (1.6)$$

with κ_k as in (1.3). There are three other kernel functions in [4] (as usual in the Eynard-Mehta setting [26]) but we will be only interested in $K_n(x_1, x_2)$. Indeed, in this paper we will express $K_n(x_1, x_2)$ in terms of a Riemann-Hilbert (RH) problem. It is an open problem to apply our method to the mixed correlation functions, which describe the interaction between the singular values of Φ_1 and Φ_2 .

The kernel (1.6) describes a well-defined determinantal point process on the positive real line for any value of $\nu > -1$. Our results on this point process will also hold for general $\nu > -1$ and have a random matrix interpretation in case ν is integer, i.e. then the particles correspond to the squared singular values of the matrix Φ_1 in (1.2). In the following we will often refer to the process described by the kernel (1.6) as the (squared) singular value process, even if ν is not an integer.

Summarizing, biorthogonal polynomials allow to obtain the correlation functions in the chiral two-matrix model. One can also find analytic expressions for the distributions of the individual singular values, see [2, 5].

In this paper we will show that the biorthogonal polynomials $P_{j,n}(x)$ and $Q_{k,n}(y)$ can be characterized as *multiple orthogonal polynomials* [33, 43] with respect to a suitable system of weight functions. Consequently we will express the kernel $K_n(x, y)$ in terms of a RH problem. This yields the chiral analogue of a result of Kuijlaars-McLaughlin [33]; see also [8, 25, 31] for some RH problems of a different nature for the non-chiral two-matrix model.

The paper [4] contains a detailed analysis for the case of linear potentials $V(x) = c_1x$, $W(y) = c_2y$. In the present paper we focus on the quadratic case

$$W(y) = y^2/2 + \alpha y, \quad \alpha \in \mathbb{R}.$$

The other potential $V(x)$ will be allowed to be an arbitrary polynomial with positive leading coefficient. Under these assumptions on V and W , we will be able to perform a Deift-Zhou asymptotic analysis of the RH problem, yielding the asymptotics of the kernel K_n . In this way we get the chiral analogues of the results by Duits-Kuijlaars-Mo [22, 23, 24, 39].

Our results imply universality in the case of a quadratic potential $W(y)$. Universality results for the chiral 1-matrix model were obtained in [3, 30, 35].

Under the additional assumption that $V(x) = x$, we give an $\alpha\tau$ -phase diagram and discuss the phase transitions. In particular the phase diagram indicates a multicritical point for the values of parameters $\alpha = -1$ and $\tau = 1$. We describe the local behavior of the singular value process near this multicritical point by means of a triple scaling limit leading to the chiral version of the main result in [20]. The new critical kernel that we obtain in the scaling limit is expressed in terms of a 4×4 matrix-valued RH problem that was introduced by one of the authors in [16] to describe a critical phenomenon for non-intersecting squared Bessel paths. Our kernel will be built from the same RH problem, but in an essentially different way than in [16].

2 Statement of results

Our first results hold in the general case where V and W are polynomial potentials with positive leading coefficients. We find it convenient to rewrite (1.4) as

$$w_n(x, y) = f_n(xy) e^{-n(V(x)+W(y))}, \quad (2.1)$$

with

$$f_n(x) := x^{\nu/2} I_\nu(2\tau n \sqrt{x}). \quad (2.2)$$

We also introduce the functions

$$h_{l,n}(x) := \int_0^\infty y^l f_n(xy) e^{-nW(y)} dy, \quad (2.3)$$

for $l \in \mathbb{N} \cup \{0\}$ and write

$$r := \deg W - 1. \quad (2.4)$$

2.1 Multiple orthogonality relations

This is our first main result.

Theorem 2.1. (*Multiple orthogonality relations*)

(a) The biorthogonal polynomial $P_{j,n}(x)$ in (1.3) is the type II multiple orthogonal polynomial with respect to the weight functions

$$e^{-nV(x)}h_{l,n}(x), \quad l = 0, \dots, 2r.$$

More precisely,

$$\int_0^\infty P_{j,n}(x)x^k e^{-nV(x)}h_{l,n}(x) dx = 0, \quad (2.5)$$

for $l = 0, \dots, 2r$ and $k = 0, \dots, \left\lfloor \frac{j-l-1}{2r+1} \right\rfloor$, where we recall the notations (2.3)–(2.4) and $\lfloor \cdot \rfloor$ denotes the integer part of a number.

(b) The polynomial $P_{j,n}(x)$ also satisfies the alternative system of multiple orthogonality relations

$$\int_0^\infty P_{j,n}(x)x^k w_{l,n}(x) dx = 0, \quad (2.6)$$

for $l = 0, \dots, 2r$ and $k = 0, \dots, \left\lfloor \frac{j-l-1}{2r+1} \right\rfloor$, with the weight functions

$$w_{l,n}(x) := \begin{cases} e^{-nV(x)}h_{l,n}(x), & l = 0, \dots, r, \\ e^{-nV(x)}x h'_{l-r-1,n}(x), & l = r+1, \dots, 2r. \end{cases} \quad (2.7)$$

Theorem 2.1 will be proved in Section 3. A similar result holds of course for the biorthogonal polynomials $Q_{j,n}(x)$.

2.2 Riemann-Hilbert problem and correlation kernel

Theorem 2.1 asserts that the polynomials $P_{j,n}$ satisfy multiple orthogonality relations of type II. Hence, they are characterized by the following RH problem [43]. In the statement of the theorem we will use the system of multiple orthogonality relations (2.6) rather than (2.5). We write $\mathbb{R}^+ := [0, \infty)$.

RH problem 2.2. We look for a $(2r+2) \times (2r+2)$ matrix-valued function $Y : \mathbb{C} \setminus \mathbb{R}^+ \rightarrow \mathbb{C}^{(2r+2) \times (2r+2)}$ satisfying the following conditions.

- (1) $Y(z)$ is analytic (entrywise) for $z \in \mathbb{C} \setminus \mathbb{R}^+$.
- (2) Y has limiting values Y_\pm on \mathbb{R}^+ , where Y_+ (Y_-) denotes the limiting value from the upper (lower) half-plane, and these limiting values satisfy the jump relation

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_{0,n}(x) & \cdots & w_{2r,n}(x) \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad (2.8)$$

for $x \in \mathbb{R}^+$, where $w_{l,n}(x)$, $l = 0, \dots, 2r$, is given in (2.7).

- (3) As $z \rightarrow \infty$, we have that

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right) \right) \text{diag}(z^n, z^{-n_0}, z^{-n_1}, \dots, z^{-n_{2r}}), \quad (2.9)$$

where $n_l = \left\lfloor \frac{n+2r-l}{2r+1} \right\rfloor$ for $l = 0, \dots, 2r$.

(4) $Y(z)$ has the following behavior near the origin:

$$\begin{aligned} Y(z) \operatorname{diag}(1, h(z)^{-1}, \dots, h(z)^{-1}) &= \mathcal{O}(1), \\ Y^{-T}(z) \operatorname{diag}(h^{-1}(z), 1, \dots, 1) &= \mathcal{O}(1), \end{aligned}$$

as $z \rightarrow 0$, $z \in \mathbb{C} \setminus \mathbb{R}^+$, where the \mathcal{O} -condition is taken to be entrywise, the superscript $-T$ denotes the inverse transpose, and

$$h(z) = \begin{cases} |z|^\nu, & \text{if } -1 < \nu < 0, \\ \log |z|, & \text{if } \nu = 0, \\ 1, & \text{if } \nu > 0. \end{cases} \quad (2.10)$$

This RH problem has a unique solution. It is constructed out of the type II multiple orthogonal polynomials and their Cauchy transforms. In particular, the $(1, 1)$ entry of Y is $P_{n,n}$. The inverse transpose Y^{-T} is given in terms of the associated type I multiple orthogonal polynomials. We refer to [43] for the details.

Condition (4) is needed to ensure that the solution of the RH problem is unique. The exact form of (4) follows from an analysis of the formulas for Y and Y^{-T} in terms of the multiple orthogonal polynomials mentioned above. This can be done as in [34, Proof of Theorem 2.4]. An essential ingredient is that $w_{l,n}(z) = \mathcal{O}(z^\nu)$, as $z \rightarrow 0$ for $l = 0, \dots, 2r$, which is immediate from (2.2) and (2.7).

The correlation kernel K_n in (1.6) has the following representation in terms of Y , see [11]

$$K_n(x, y) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} 0 & w_{0,n}(y) & w_{1,n}(y) & \cdots & w_{2r,n}(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T, \quad (2.11)$$

where the superscript T denotes the transpose and both row vectors have length $2r + 2$.

This representation allows us to derive the large n limit of the correlation kernel in the case of a quadratic potential $W(y) = y^2/2 + \alpha y$ by performing a Deift/Zhou steepest descent analysis on RH problem 2.2. As will be clear from the analysis, this corresponds to a quartic potential in the non-chiral two-matrix model studied by Duits-Kuijlaars-Mo in [22, 23, 39]. We will largely follow the line of thought in these works, however, at several places complications will arise.

The key to the steepest descent analysis is a third order differential equation for $h_{0,n}(x)$ from (2.3), to be stated in Section 4. It plays the same role as the Pearcey equation in [22, 23] but is considerably more complicated. The steepest descent analysis itself will be described in detail in Section 5.

2.3 Vector equilibrium problem and connection with the non-chiral two-matrix model

From this point we will assume that the second potential is quadratic

$$W(y) = \frac{y^2}{2} + \alpha y, \quad \alpha \in \mathbb{R}, \quad (2.12)$$

and that V is a polynomial with a positive leading coefficient. As one of our main results we will characterize the limiting mean squared singular value distribution of Φ_1 in this case using a vector equilibrium problem. More precisely, it is the first measure of a triplet of measures minimizing an energy functional under certain conditions. Before we introduce this vector equilibrium problem we review a related equilibrium problem that arises in the study of the non-chiral two-matrix model.

Vector equilibrium problem for the non-chiral two-matrix model

In the recent paper [23], Duits, Kuijlaars, and Mo consider a Hermitian two-matrix model of the form

$$\frac{1}{Z_n^{\text{NC}}} \exp \left(-n \text{Tr} (V^{\text{NC}}(M_1) + W^{\text{NC}}(M_2) - \tau M_1 M_2) \right) dM_1 dM_2, \quad (2.13)$$

defined on pairs (M_1, M_2) of $n \times n$ Hermitian matrices. Here, Z_n^{NC} is a normalization constant, V^{NC} is a general even polynomial with a positive leading coefficient, W^{NC} is a quartic polynomial given by

$$W^{\text{NC}}(y) = \frac{1}{4}y^4 + \frac{\alpha}{2}y^2, \quad \alpha \in \mathbb{R}, \quad (2.14)$$

and $\tau > 0$ is the coupling constant. Throughout this paper, we use the superscript ^{NC} to distinguish functions and constants related to the non-chiral two-matrix model from similar notions in the chiral two-matrix model.

We define, as usual (see [41]), the logarithmic energy of a measure ν by

$$I(\nu) = \iint \log \frac{1}{|x - y|} d\nu(x) d\nu(y), \quad (2.15)$$

and the mutual energy of two measures ν_1, ν_2 by

$$I(\nu_1, \nu_2) = \iint \log \frac{1}{|x - y|} d\nu_1(x) d\nu_2(y). \quad (2.16)$$

The main result of [23] is that the limiting mean eigenvalue distribution of M_1 can be described by the first component of a triplet $(\mu_1^{\text{NC}}, \mu_2^{\text{NC}}, \mu_3^{\text{NC}})$, which are three measures minimizing the energy functional

$$E^{\text{NC}}(\nu_1, \nu_2, \nu_3) := \sum_{j=1}^3 I(\nu_j) - \sum_{j=1}^2 I(\nu_j, \nu_{j+1}) + \int V_1^{\text{NC}}(x) d\nu_1(x) + \int V_3^{\text{NC}}(x) d\nu_3(x), \quad (2.17)$$

with V_1^{NC} and V_3^{NC} being certain symmetric external fields on \mathbb{R} , and where the minimization is among all positive measures such that

- (a) ν_1 is a measure on \mathbb{R} with total mass 1;
- (b) ν_2 is a measure on $i\mathbb{R}$ with total mass $2/3$ that satisfies the constraint

$$\nu_2 \leq \sigma_2^{\text{NC}}, \quad (2.18)$$

where σ_2^{NC} is a certain positive symmetric measure on the imaginary axis;

- (c) ν_3 is a measure on \mathbb{R} with total mass $1/3$;
- (d) $I(\nu_j) < \infty$ for $j = 1, 2, 3$.

This equilibrium problem clearly depends on the input data V_1^{NC} , V_3^{NC} , and σ_2^{NC} . For the exact definitions of these notions we refer to [23]. The unique solvability of the vector equilibrium problem follows from [23, 29].

Vector equilibrium problem for the chiral two-matrix model

Given the triplet (V, W, τ) defining the chiral two-matrix model, we define

$$V^{\text{NC}}(x) = \frac{1}{2}V(x^2) \quad \text{and} \quad W^{\text{NC}}(y) = \frac{1}{2}W(y^2). \quad (2.19)$$

Note that this definition is consistent with (2.14) and (2.12). The triplet $(V^{\text{NC}}, W^{\text{NC}}, \tau)$ then characterizes an associated non-chiral two-matrix model.

The vector equilibrium problem that is appropriate in our chiral setting is a ‘squared’ version of the above vector equilibrium problem for the associated non-chiral two-matrix model. More precisely, we consider the energy functional

$$E(\nu_1, \nu_2, \nu_3) := \sum_{j=1}^3 I(\nu_j) - \sum_{j=1}^2 I(\nu_j, \nu_{j+1}) + \int V_1(x) d\nu_1(x) + \int V_3(x) d\nu_3(x), \quad (2.20)$$

and define the input data

$$V_1(x) := 2V_1^{\text{NC}}(\sqrt{x}), \quad V_3(x) := 2V_3^{\text{NC}}(\sqrt{x}), \quad d\sigma_2(x) := 2d\sigma_2^{\text{NC}}(i\sqrt{-x}). \quad (2.21)$$

Then V_1 and V_3 are external fields on \mathbb{R}^+ and σ_2 is a positive measure on \mathbb{R}^- . The vector equilibrium problem is then to minimize $E(\nu_1, \nu_2, \nu_3)$ among all positive measures ν_1, ν_2 and ν_3 satisfying the following conditions.

- (a) ν_1 is a measure on \mathbb{R}^+ with total mass 1.
- (b) ν_2 is a measure on $\mathbb{R}^- := (-\infty, 0]$ with total mass $2/3$ such that

$$\nu_2 \leq \sigma_2, \quad (2.22)$$

where σ_2 is defined in (2.21).

- (c) ν_3 is a measure on \mathbb{R}^+ with total mass $1/3$.
- (d) $I(\nu_j) < \infty$ for $j = 1, 2, 3$.

This vector equilibrium problem has a unique solution described in the following theorem. We denote the support of a measure ν by $S(\nu)$.

Theorem 2.3. *The equilibrium problem (2.20)–(2.22) has a unique solution (μ_1, μ_2, μ_3) . Moreover, if $0 \notin S(\mu_1)$ or $0 \notin S(\sigma_2 - \mu_2)$ then*

$$S(\mu_1) = \bigcup_{j=1}^N [a_j, b_j],$$

for some $N \in \mathbb{N}$ and $0 \leq a_1 < b_1 < a_2 < \dots < a_N < b_N$ and on each of the intervals $[a_j, b_j]$ in $S(\mu_1)$ there is a density

$$\frac{d\mu_1}{dx} = \rho_1(x) = \begin{cases} \frac{1}{\pi} g_j(x) \sqrt{(b_j - x)(x - a_j)}, & x \in [a_j, b_j], \quad \text{if } a_j > 0, \\ \frac{1}{\pi} g_1(x) \sqrt{(b_1 - x)x^{-1}}, & x \in [0, b_1], \quad \text{if } a_1 = 0, \end{cases}$$

where g_j is nonnegative and real analytic on $[a_j, b_j]$.

Proof. We claim that

$$(\mathrm{d}\mu_1(x), \mathrm{d}\mu_2(x), \mathrm{d}\mu_3(x)) := (2\mathrm{d}\mu_1^{\mathrm{NC}}(\sqrt{x}), 2\mathrm{d}\mu_2^{\mathrm{NC}}(\sqrt{x}), 2\mathrm{d}\mu_3^{\mathrm{NC}}(\sqrt{x})), \quad (2.23)$$

where the right hand side denotes the solution to the equilibrium problem (2.17)–(2.18) with the same parameters τ, α . To prove (2.23), let us denote for any measure μ symmetric with respect to the origin, the squared measure $\hat{\mu}$ by the rule $\mathrm{d}\hat{\mu}(x) = 2\mathrm{d}\mu(\sqrt{x})$. Since μ is symmetric we do not have to precise the branch cut of the square root. Thus if μ has a density $\mathrm{d}\mu(x) = \rho(x)\mathrm{d}x$ then

$$\mathrm{d}\hat{\mu}(x) := 2\rho(\sqrt{x})\mathrm{d}\sqrt{x} = \frac{\rho(\sqrt{x})}{\sqrt{x}}\mathrm{d}x.$$

Now the mutual energies of two measures ν_1, ν_2 and the corresponding squared measures $\hat{\nu}_1, \hat{\nu}_2$ are related by

$$I(\hat{\nu}_1, \hat{\nu}_2) = 2I(\nu_1, \nu_2),$$

see e.g. [41, Th. IV.1.10(f)]. With the help of this relation, and using (2.21), (2.14) and (2.12), the claimed relation (2.23) follows.

Given this, the theorem is a corollary of [23, Theorem 1.1]. \square

Remark 2.4. If $a_1 = 0$ the density of μ_1 blows up like an inverse square root at the origin as is the case for the Marchenko-Pastur density [37].

2.4 Classification into cases

In this model we distinguish a number of regular and singular cases depending on the supports and densities of the measures μ_1 , $\sigma_2 - \mu_2$, and μ_3 . The classification in our chiral setting is inherited from the classification for the associated non-chiral models given in [23, Section 1.5], e.g. we say that our chiral model belongs to Case I if the associated non-chiral model is in Case I according to [23, Section 1.5]. This leads us to distinguish five generic cases and 8 singular cases. The classification of the five generic phases depends on whether 0 is in the support of the measures μ_1 , $\sigma_2 - \mu_2$, and μ_3 or not. We have the following cases with regular behavior of the supports at zero:

Case I $0 \in S(\mu_1)$, $0 \notin S(\sigma_2 - \mu_2)$ and $0 \in S(\mu_3)$,

Case II $0 \notin S(\mu_1)$, $0 \notin S(\sigma_2 - \mu_2)$ and $0 \in S(\mu_3)$,

Case III $0 \notin S(\mu_1)$, $0 \in S(\sigma_2 - \mu_2)$ and $0 \notin S(\mu_3)$,

Case IV $0 \in S(\mu_1)$, $0 \notin S(\sigma_2 - \mu_2)$ and $0 \notin S(\mu_3)$,

Case V $0 \notin S(\mu_1)$, $0 \notin S(\sigma_2 - \mu_2)$ and $0 \notin S(\mu_3)$.

We discuss only three of the singular cases. For the remaining five singular cases we refer to [23]. The critical phenomena corresponding to these remaining five cases can already be found in the one-matrix model, so we will not discuss them here.

Singular supports I $0 \in S(\mu_1) \cap S(\sigma_2 - \mu_2)$, $0 \notin S(\mu_3)$.

Singular supports II $0 \notin S(\mu_1)$, $0 \in S(\sigma_2 - \mu_2) \cap S(\mu_3)$.

Singular supports III $0 \in S(\mu_1) \cap S(\sigma_2 - \mu_2) \cap S(\mu_3)$.

The last one is a multi-singular case.

Except for Case V, all these phenomena already occur in the simplest version of the model in which the potential $V(x) = x$. For this situation we will establish a phase diagram in Section 2.6.

The next theorem will only be proved in the regular cases that we define as follows.

Definition 2.5. The triplet (V, W, τ) is regular if the associated triplet $(V^{\text{NC}}, W^{\text{NC}}, \tau)$ is regular in the sense of [23, Definition 1.3], with V^{NC} and W^{NC} as in (2.19).

In particular for a regular triplet it holds that $S(\mu_1) \cap S(\sigma_2 - \mu_2) = \emptyset$ and $S(\mu_3) \cap S(\sigma_2 - \mu_2) = \emptyset$. Moreover the functions g_j , $j = 1, \dots, N$, in Theorem 2.3 are nonzero for all $x \in [a_j, b_j]$, i.e. the density of the measure μ_1 does not vanish in the interior of the support and behaves like a square root at the nonzero endpoints of the support. The same holds for the densities of $\sigma_2 - \mu_2$ and μ_3 . There is an extra condition on the variational inequality for μ_1 which guarantees that no extra interval emerges in the support of μ_1 when continuously varying the potentials.

2.5 Limiting mean singular value distribution

The measure μ_1 is the limiting mean squared singular value distribution of the matrix Φ_1 in the chiral two-matrix model as $n \rightarrow \infty$. This statement holds for W as in (2.12) and general V and τ , but we will only prove it for the regular cases, see Definition 2.5

Theorem 2.6. *Suppose (V, W, τ) is regular. Let μ_1 be the first component of the minimizer (μ_1, μ_2, μ_3) of the vector equilibrium problem (2.20)–(2.22). Then μ_1 is the limiting mean particle distribution of the determinantal point process with correlation kernel (1.6) as $n \rightarrow \infty$ with $n \equiv 0 \pmod{3}$.*

This theorem is equivalent to the statement

$$\lim_{n \rightarrow \infty} \frac{1}{n} K_n(x, x) = \rho_1(x) = \frac{d\mu_1}{dx}(x), \quad x > 0, \quad n \equiv 0 \pmod{3}.$$

The proof is given in Section 5.8. It is based on a lengthy Deift/Zhou steepest descent analysis carried out in Section 5. Without too much extra effort one could also obtain the universal scaling limits that are typical for unitary random matrix ensembles. More precisely, if $\rho_1(x^*) > 0$, where ρ_1 denotes the density of the measure μ_1 , we retrieve the sine kernel as a scaling limit

$$\lim_{n \rightarrow \infty} K_n \left(x^* + \frac{x}{n\rho_1(x^*)}, x^* + \frac{y}{n\rho_1(x^*)} \right) = \frac{\sin \pi(x - y)}{\pi(x - y)}.$$

If x^* is a nonzero endpoint of $S(\mu_1)$, i.e. $x^* \in \{a_1, b_1, \dots, a_N, b_N\} \setminus \{0\}$, then

$$\lim_{n \rightarrow \infty} \frac{1}{(cn)^{2/3}} K_n \left(x^* \pm \frac{x}{(cn)^{2/3}}, x^* \pm \frac{y}{(cn)^{2/3}} \right) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y},$$

where we choose the $+$ -sign ($-$ -sign) if x^* is a left (right) endpoint of the support of μ_1 . Recall that we are in the regular case so that the density vanishes like a square root at $x^* > 0$. However, if $x^* = a_1 = 0$ then the density blows up as an inverse square root and we would obtain the Bessel kernel of order ν as a scaling limit

$$\lim_{n \rightarrow \infty} \frac{1}{(cn)^2} K_n \left(\frac{x}{(cn)^2}, \frac{y}{(cn)^2} \right) = \frac{J_\nu(\sqrt{x})\sqrt{y}J'_\nu(\sqrt{y}) - \sqrt{x}J'_\nu(\sqrt{x})J_\nu(\sqrt{y})}{2(x - y)},$$

for $x, y > 0$ and a suitable constant $c > 0$. We will not discuss this any further.

2.6 Phase diagram in the quadratic/linear case

From here we restrict ourselves to the very specific model of quadratic and linear potentials

$$V(x) = x, \quad W(y) = \frac{y^2}{2} + \alpha y, \quad \alpha \in \mathbb{R}. \quad (2.24)$$

For this concrete model we can construct a phase diagram, i.e. determine which values of (α, τ) correspond to which case of the classification in Section 2.4. It turns out that the case ‘Singular

supports III' occurs and this will get most of our attention. To establish the phase diagram, we first discuss the behavior of the supports of the measures μ_1 , $\sigma_2 - \mu_2$, and μ_3 and how they depend on α and τ . It follows from (2.23) and [21] that the supports of the measures μ_1 , $\sigma_2 - \mu_2$, and μ_3 have the following form

$$\begin{aligned}\text{supp}(\mu_1) &= [\beta_1, \beta_0], \\ \text{supp}(\sigma_2 - \mu_2) &= (-\infty, -\beta_2], \\ \text{supp}(\mu_3) &= [\beta_3, \infty),\end{aligned}$$

for some $\beta_0 > \beta_1 \geq 0$, $\beta_2, \beta_3 \geq 0$ that all depend on the values of $\alpha \in \mathbb{R}$ and $\tau > 0$. We distinguish a number of cases, depending on whether β_1 , β_2 , or β_3 are equal to zero, or not. At least one of these is zero, and generically, no two consecutive ones are zero. According to the classification in Section 2.4 we have

Case I: $\beta_1 = 0$, $\beta_2 > 0$, and $\beta_3 = 0$.

Case II: $\beta_1 > 0$, $\beta_2 > 0$, and $\beta_3 = 0$.

Case III: $\beta_1 > 0$, $\beta_2 = 0$, and $\beta_3 > 0$.

Case IV: $\beta_1 = 0$, $\beta_2 > 0$, and $\beta_3 > 0$.

Case V does not occur in this specific model.

The phase diagram in Figure 1 shows which values of (α, τ) correspond to these four cases. The different cases are separated by the curves given by the equations

$$\tau = \sqrt{\alpha + 2}, \quad -2 \leq \alpha < \infty, \quad \text{and} \quad \tau = \sqrt{-\frac{1}{\alpha}}, \quad -\infty < \alpha < 0.$$

On these critical curves two of the numbers β_1, β_2 and β_3 are equal to zero. For example, on the curve between Case III and Case IV, we have $\beta_1 = \beta_2 = 0$, while $\beta_3 > 0$. Finally, note the multi-critical point $(\alpha, \tau) = (-1, 1)$ in the phase diagram, where $\beta_1 = \beta_2 = \beta_3 = 0$. All four cases come together at this point in the (α, τ) -plane. The nature of this multi-critical point is discussed in the next section.

We do not study the other types of critical behavior in this paper, but we shortly comment on them now. The geometry of the supports of μ_1 , $\sigma_2 - \mu_2$, and μ_3 suggests that the transition on the curve $\tau = \sqrt{\alpha + 2}$, $\alpha > -1$, is described by the inhomogeneous Painlevé II kernel as in [10]. The transition on the curve $\tau = \sqrt{-1/\alpha}$, $\alpha < -1$, on the other hand is described by the chiral version of the Pearcey kernel as in [32]. The transitions on the dashed lines in Figure 1 are situated on the non-physical sheets of the underlying Riemann surface and, hence, do not affect the local correlations of the singular values of Φ_1 .

The curve $\tau = \sqrt{\alpha + 2}$, $\alpha > -2$ is also critical for the singular values of Φ_2 . The other curve is not critical in that context.

Figure 2 shows the phase diagram again together with the shape of the limiting mean squared singular value densities above, below, and on the critical curve.

2.7 A triple scaling limit

Let us now focus on the squared singular value process of Φ_1 near the critical parameters $\tau = 1$ and $\alpha = -1$, by means of a triple scaling limit. To this end, we rescale α and τ near the critical values in the following way

$$\begin{pmatrix} \alpha \\ \tau \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + an^{-1/3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + bn^{-2/3} \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad (2.25)$$

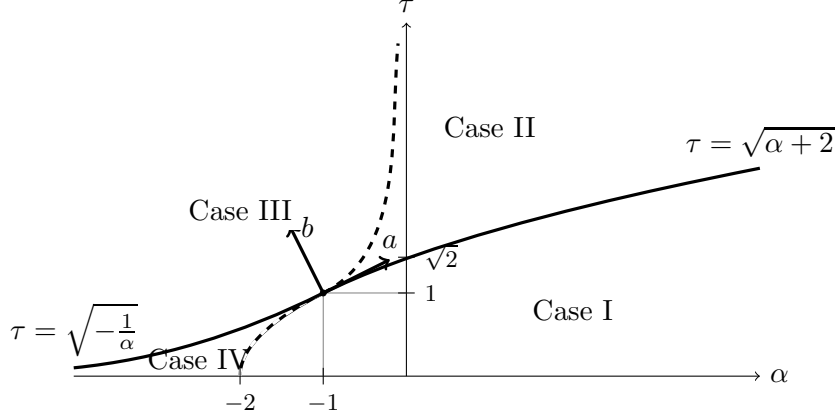


Figure 1: The phase diagram in the $\alpha\tau$ -plane: the critical curves $\tau = \sqrt{\alpha + 2}$ and $\tau = \sqrt{-\frac{1}{\alpha}}$ separate the four cases. The cases are distinguished by the fact whether 0 is in the support of the measures μ_1 , $\sigma_2 - \mu_2$, and μ_3 , or not.

for $a, b \in \mathbb{R}$. We also scale the space variables with

$$x = un^{-4/3}, \quad \text{and} \quad y = vn^{-4/3}, \quad u, v > 0$$

and compute the limiting behavior of $K_n(x, y)$ as $n \rightarrow \infty$.

The limiting kernel is characterized by the solution to the following RH problem introduced in [16]. The RH problem has jumps on a contour in the complex plane consisting of 10 rays emanating from the origin. More precisely, we fix two numbers φ_1, φ_2 such that $0 < \varphi_1 < \varphi_2 < \pi/2$ and define the half-lines Γ_k , $k = 0, \dots, 9$, by

$$\Gamma_0 = \mathbb{R}^+, \quad \Gamma_1 = e^{i\varphi_1}\mathbb{R}^+, \quad \Gamma_2 = e^{i\varphi_2}\mathbb{R}^+, \quad \Gamma_3 = e^{i(\pi-\varphi_2)}\mathbb{R}^+, \quad \Gamma_4 = e^{i(\pi-\varphi_1)}\mathbb{R}^+, \quad (2.26)$$

and

$$\Gamma_{5+k} = -\Gamma_k, \quad k = 0, \dots, 4. \quad (2.27)$$

All rays Γ_k , $k = 0, \dots, 9$, are oriented towards infinity, as shown in Figure 3. We also denote by Ω_k the region in \mathbb{C} which lies between the rays Γ_k and Γ_{k+1} , for $k = 0, \dots, 9$, where we identify $\Gamma_{10} := \Gamma_0$. Now we can state the RH problem.

RH problem 2.7. We look for a 4×4 matrix-valued function $M : \mathbb{C} \setminus \left(\bigcup_{k=0}^9 \Gamma_k\right) \rightarrow \mathbb{C}^{4 \times 4}$ (which also depends parametrically on $\tilde{\nu} > -1/2$ and on the complex parameters $r_1, r_2, s, t \in \mathbb{C}$) satisfying the following conditions.

(1) $M(\zeta)$ is analytic (entrywise) for $\zeta \in \mathbb{C} \setminus \left(\bigcup_{k=0}^9 \Gamma_k\right)$.

(2) For $\zeta \in \Gamma_k$, the limiting values

$$M_+(\zeta) = \lim_{\substack{z \rightarrow \zeta \\ z \text{ on } +\text{-side of } \Gamma_k}} M(z), \quad M_-(\zeta) = \lim_{\substack{z \rightarrow \zeta \\ z \text{ on } -\text{-side of } \Gamma_k}} M(z)$$

exist, where the $+$ -side and $-$ -side of Γ_k are the sides which lie on the left and right of Γ_k , respectively, when traversing Γ_k according to its orientation. These limiting values satisfy the jump relation

$$M_+(\zeta) = M_-(\zeta)J_k(\zeta), \quad k = 0, \dots, 9,$$

where the jump matrix $J_k(\zeta)$ for each ray Γ_k is shown in Figure 3.

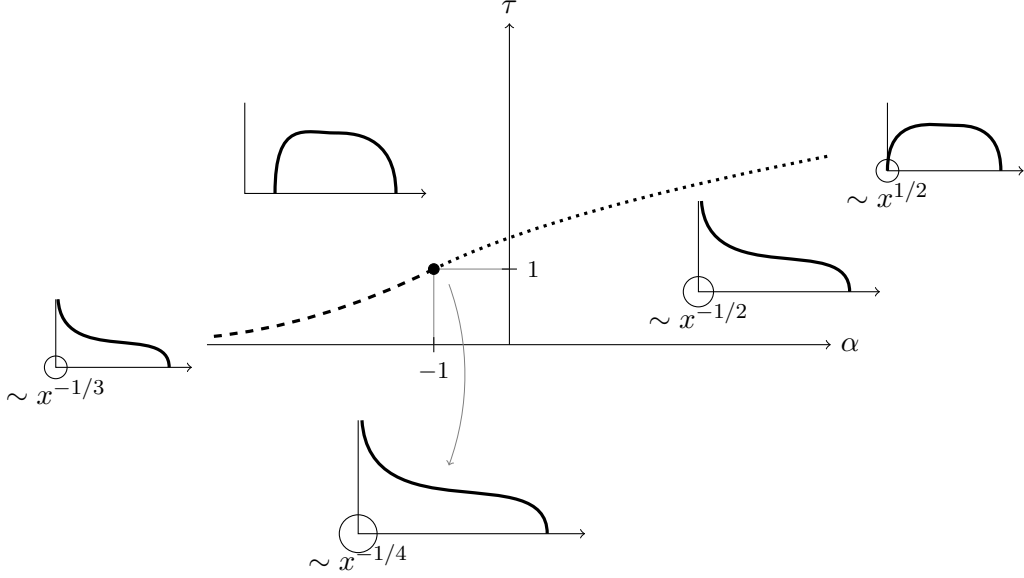


Figure 2: The phase diagram in the $\alpha\tau$ -plane with density plots. Below the critical curve the limiting mean squared singular value density of Φ_1 is supported on one interval touching the origin. At the origin the density blows up like an inverse square root. Above this line the singular values cluster on an interval away from zero. On the critical curve the transition between these two regimes occurs. There the density is supported on an interval including zero. On the part of critical curve to the right of the point $(-1, 1)$ (dotted) the density vanishes like a square root. To the left of this point (dashed) the exponent at the origin is $-1/3$. At the point $(-1, 1)$ we have a transition between these two types of critical behavior. There the density blows up with an exponent $-1/4$.

(3) As $\zeta \rightarrow \infty$ we have

$$M(\zeta) = \left(I + \frac{M_1}{\zeta} + \frac{M_2}{\zeta^2} + O\left(\frac{1}{\zeta^3}\right) \right) \text{diag}((-\zeta)^{-1/4}, \zeta^{-1/4}, (-\zeta)^{1/4}, \zeta^{1/4}) \\ \times A \text{diag}\left(e^{-\psi_2(\zeta)+t\zeta}, e^{-\psi_1(\zeta)-t\zeta}, e^{\psi_2(\zeta)+t\zeta}, e^{\psi_1(\zeta)-t\zeta}\right)$$

where the coefficient matrices M_1, M_2, \dots depend on the parameters $\tilde{\nu}, r_1, r_2, s$ and t , but not on ζ , and where we define

$$A := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & i \\ -i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}, \quad (2.28)$$

$$\psi_1(\zeta) = \frac{2}{3}r_1\zeta^{3/2} + 2s\zeta^{1/2}, \quad \psi_2(\zeta) = \frac{2}{3}r_2(-\zeta)^{3/2} + 2s(-\zeta)^{1/2}. \quad (2.29)$$

(4) $M(\zeta)$ behaves for $\zeta \rightarrow 0$ as

$$M(\zeta) = \mathcal{O}(\zeta^{\tilde{\nu}}), \quad M^{-1}(\zeta) = \mathcal{O}(\zeta^{\tilde{\nu}}), \quad \text{if } \tilde{\nu} \leq 0,$$

and

$$\begin{cases} M(\zeta) \text{diag}(\zeta^{-\tilde{\nu}}, \zeta^{\tilde{\nu}}, \zeta^{\tilde{\nu}}, \zeta^{-\tilde{\nu}}) = \mathcal{O}(1), & \zeta \in \Omega_1 \cup \Omega_8, \\ M(\zeta) \text{diag}(\zeta^{\tilde{\nu}}, \zeta^{-\tilde{\nu}}, \zeta^{-\tilde{\nu}}, \zeta^{\tilde{\nu}}) = \mathcal{O}(1), & \zeta \in \Omega_3 \cup \Omega_6, \\ M(\zeta) = \mathcal{O}(\zeta^{-\tilde{\nu}}), & \zeta \notin (\Omega_1 \cup \Omega_3 \cup \Omega_6 \cup \Omega_8), \end{cases} \quad \text{if } \tilde{\nu} \geq 0.$$

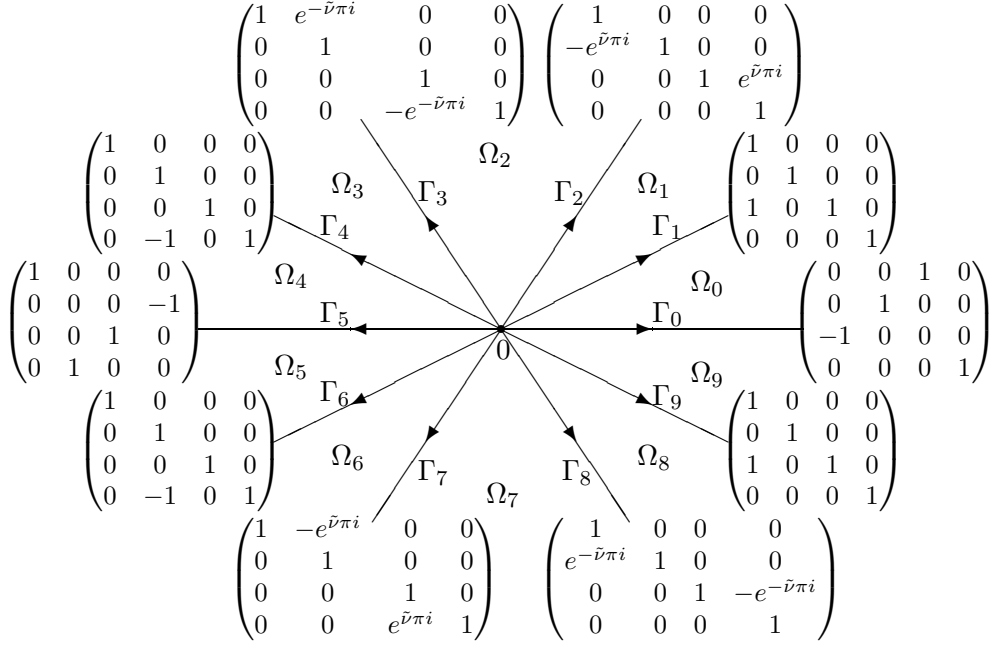


Figure 3: The figure shows the jump contours Γ_k in the complex ζ -plane and the corresponding jump matrix J_k on Γ_k , $k = 0, \dots, 9$, in the RH problem for $M = M(\zeta)$. We denote by Ω_k the region between the rays Γ_k and Γ_{k+1} .

It was proved in [16] that this RH problem is solvable for $r_1 = r_2 > 0$ and $s, t \in \mathbb{R}$. In the same paper this RH problem is shown to be related to the Hastings-McLeod solution of the inhomogeneous Painlevé II equation $q''(x) = xq(x) + 2q^3(x) - \tilde{\nu} - 1/2$.

We transform the RH matrix $M(z)$ into a new matrix $\widehat{M}(z)$ as follows

$$\widehat{M}(z) := \text{diag}(z^{1/4}, z^{-1/4}, z^{1/4}, z^{-1/4}) \text{diag} \left(\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right) M(iz^{1/2}). \quad (2.30)$$

The transformed matrix $\widehat{M}(z)$ depends on the same parameters $r_1, r_2, s, t, \tilde{\nu}$ as $M(z)$. The matrix $\widehat{M}(z)$ satisfies a RH problem by itself but we will not state it here.

For $u, v > 0$, we now define the critical kernel $K^{cr}(u, v; s, t, \nu)$ by

$$K^{cr}(u, v; s, t, \nu) = \frac{1}{2\pi i(u-v)} \begin{pmatrix} 0 & 0 & -ie^{-\nu\pi i} & 1 \end{pmatrix} \widehat{M}(v; s, t, \tilde{\nu})^{-1} \\ \times \widehat{M}(u; s, t, \tilde{\nu}) \begin{pmatrix} 0 & 0 & -ie^{\nu\pi i} & 1 \end{pmatrix}^T, \quad (2.31)$$

where $\widehat{M}(z; s, t, \tilde{\nu})$ is defined by (2.30) with $M(\zeta) = M(\zeta; s, t, \tilde{\nu})$ the unique solution to RH problem 2.7 with parameters

$$\begin{cases} r_1 = r_2 = 2, \\ s, t \in \mathbb{R}, \\ \tilde{\nu} = \nu + 1/2. \end{cases} \quad (2.32)$$

It can be shown as in [20, Appendix A] that this kernel is different from the hard-edge tacnode kernel discovered in [16].

We can then state our final main result.

Theorem 2.8. Assume $\nu > -1$ and let K_n be the kernel in (1.6) with V and W as in (2.24). If ν is integer this kernel describes the squared singular values of M_1 when averaged over M_2 .

Set

$$\begin{pmatrix} \alpha \\ \tau \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + an^{-1/3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + bn^{-2/3} \begin{pmatrix} -1 \\ 2 \end{pmatrix},$$

for $a, b \in \mathbb{R}$. Then for $n \rightarrow \infty$ and $n \equiv 0 \pmod{3}$

$$\lim_{n \rightarrow \infty} \frac{1}{n^{4/3}} K_n \left(\frac{u}{n^{4/3}}, \frac{v}{n^{4/3}} \right) = \frac{v^{\nu/2}}{u^{\nu/2}} K^{\text{cr}} \left(u, v; \frac{1}{2}(a^2 - 5b), 2a, \nu \right), \quad (2.33)$$

uniformly for u, v in compact subsets of \mathbb{R}^+ .

This theorem will be proved in Section 6.8. The prefactor $v^{\nu/2}/u^{\nu/2}$ has no influence on the singular value correlations. The phase diagram in Figure 1 suggests that by taking proper scaling limits, this critical kernel converges to the inhomogeneous Painlevé II kernel or to the kernel in [32]. The non-chiral version of this statement has been shown in [20, 28].

2.8 Organization of the paper

The rest of this paper is organized as follows. In Section 3 we prove Theorem 2.1 on the multiple orthogonality relations for the biorthogonal polynomials $P_{j,n}(x)$ and $Q_{k,n}(y)$. In Section 4 we obtain and analyze a third-order differential equation related to the chiral two-matrix model in the case of quadratic $W(y) = y^2/2 + \alpha y$. In Section 5 we perform a steepest descent analysis on RH problem 2.2 for quadratic W leading to the proof of Theorem 2.6. In Section 6 we adapt the steepest descent analysis of the preceding section to prove Theorem 2.8. The main difference is in the local parametrix at the origin which is now built using the solution to RH problem 2.7.

3 Proof of Theorem 2.1

3.1 Preliminary lemma

Lemma 3.1. *The function $f_n(x)$ in (2.2) satisfies the differential equation*

$$x f_n''(x) - (\nu - 1) f_n'(x) = (\tau n)^2 f_n(x). \quad (3.1)$$

Proof. This is immediate from the definition (2.2) and the fact that $I_\nu(z)$ satisfies the modified Bessel equation [1]

$$z^2 I_\nu''(z) + z I_\nu'(z) - (z^2 + \nu^2) I_\nu(z) = 0. \quad (3.2)$$

□

3.2 Proof of Theorem 2.1(a)

We claim that for any polynomial P and any nonnegative integer l , one has

$$\int_0^\infty x^k P(x) e^{-nV(x)} \int_0^\infty y^l f_n(xy) e^{-nW(y)} dy dx = \int_0^\infty \int_0^\infty P(x) \pi_{k,l}(y) w_n(x, y) dx dy, \quad (3.3)$$

where $\pi_{k,l}$ is a polynomial of exact degree $k(2r+1) + l$.

We prove the claim by induction on k . The case $k = 0$ is trivial. Let us assume that the

statement is valid up to k and compute the left-hand side of (3.3) for $k + 1$

$$\begin{aligned}
& \int_0^\infty x^{k+1} P(x) e^{-nV(x)} \int_0^\infty y^l f_n(xy) e^{-nW(y)} dy dx \\
&= \frac{1}{(\tau n)^2} \int_0^\infty x^k P(x) e^{-nV(x)} \int_0^\infty y^{l+1} x^2 f_n''(xy) e^{-nW(y)} dy dx \\
&\quad - \frac{\nu - 1}{(\tau n)^2} \int_0^\infty x^k P(x) e^{-nV(x)} \int_0^\infty y^l x f_n'(xy) e^{-nW(y)} dy dx \\
&= \frac{1}{(\tau n)^2} \int_0^\infty x^k P(x) e^{-nV(x)} \int_0^\infty \tilde{\pi}_{l+1+2r}(y) f_n(xy) e^{-nW(y)} dy dx \\
&\quad + \frac{\nu - 1}{(\tau n)^2} \int_0^\infty x^k P(x) e^{-nV(x)} \int_0^\infty \tilde{\pi}_{l+r}(y) f_n(xy) e^{-nW(y)} dy dx,
\end{aligned}$$

where $\tilde{\pi}_i$ denotes a polynomial of degree i . We used (3.1) for the first equality. The second equality is based on integration by parts. Note that the integrated terms vanish for $\nu > 0$ and cancel each other out for $-1 < \nu \leq 0$. Now expanding

$$\frac{1}{(\tau n)^2} \tilde{\pi}_{l+1+2r}(y) + \frac{\nu - 1}{(\tau n)^2} \tilde{\pi}_{l+r}(y) = \sum_{i=0}^{l+1+2r} a_i y^i,$$

and applying the induction hypothesis yields

$$\begin{aligned}
& \int_0^\infty x^{k+1} P(x) e^{-nV(x)} \int_0^\infty y^l f_n(xy) e^{-nW(y)} dy dx \\
&= \sum_{i=0}^{l+1+2r} a_i \int_0^\infty x^k P(x) e^{-nV(x)} \int_0^\infty y^i f_n(xy) e^{-nW(y)} dy dx \\
&= \sum_{i=0}^{l+1+2r} a_i \int_0^\infty \int_0^\infty P(x) \pi_{k,i}(y) w_n(x, y) dy dx \\
&= \int_0^\infty \int_0^\infty P(x) \pi_{k+1,l}(y) w_n(x, y) dy dx.
\end{aligned}$$

Here $\pi_{k+1,l}$ is defined as

$$\sum_{i=0}^{l+1+2r} a_i \pi_{k,i},$$

and is, therefore, of degree $(k + 1)(2r + 1) + l$ as it should be. This proves the claim.

To show (2.5), note that the set $A = \{\pi_{k,l} \mid k = 0, 1, 2, 3, \dots, l = 0, \dots, 2r\}$ spans the space of polynomials. If we take $P(x) = P_{j,n}(x)$ then the right-hand side of (3.3) vanishes for any $\pi_{k,l} \in A$ satisfying $k(2r + 1) + l < j$. This leads to vanishing of the left-hand side of (3.3) for every $k = 0, \dots, \left\lfloor \frac{j-l-1}{2r+1} \right\rfloor$, $l = 0, \dots, 2r$. Hence, $P_{j,n}$ is the j -th multiple orthogonal polynomial satisfying (2.5). This proves Theorem 2.1(a).

3.3 Proof of Theorem 2.1(b)

To prove (2.6), it is sufficient to establish linear relations between the functions $h_{j,n}$ and their derivatives.

Lemma 3.2. *The function $xh'_{l,n}(x)$, see (2.3) for $h_{l,n}$, can be written as a linear combination of the functions $h_{l+j,n}(x)$, $j = 0, \dots, r + 1$,*

$$xh'_{l,n}(x) = -(l + 1)h_{l,n}(x) + n \sum_{j=1}^{r+1} c_j h_{l+j,n}(x), \tag{3.4}$$

for any $l \in \mathbb{N} \cup \{0\}$, where we write the polynomial W in (2.1) as

$$W(y) = \sum_{j=1}^{r+1} \frac{c_j}{j} y^j, \quad (3.5)$$

with coefficients $c_k \in \mathbb{R}$, $c_{r+1} > 0$. Here we assume without loss of generality that the constant term of W vanishes.

Proof. A straightforward calculation gives

$$\begin{aligned} xh'_{l,n}(x) &:= x \int_0^\infty y^l \frac{\partial}{\partial x} (f_n(xy)) e^{-nW(y)} dy \\ &= \int_0^\infty y^{l+1} \frac{\partial}{\partial y} (f_n(xy)) e^{-nW(y)} dy \\ &= - \int_0^\infty f_n(xy) \frac{\partial}{\partial y} (y^{l+1} e^{-nW(y)}) dy \\ &= -(l+1)h_{l,n}(x) + n \sum_{j=1}^{r+1} c_j h_{l+j,n}. \end{aligned}$$

Here the first step follows from the definition (2.3), the second step is obvious by symmetry, the third step is integration by parts, and the last step uses (3.5). \square

The above lemma then implies the alternative system of weight functions for the multiple orthogonal polynomials $P_{j,n}(x)$ in (2.6). This proves Theorem 2.1(b).

4 Differential equation for $h_{0,n}(x)$ with quadratic W

From here we assume again that W is quadratic and depends on the real parameter α as in (2.12). Then $r = \deg W - 1 = 1$.

In this section we obtain a third-order differential equation for the function

$$h_{0,n}(x) = \int_0^\infty f_n(xy) e^{-nW(y)} dy, \quad (4.1)$$

see (2.3) and (2.2). We will study this differential equation in detail and construct a Wronskian matrix from three special solutions to the equation. We will show that this Wronskian matrix satisfies a certain 3×3 RH problem, which should be regarded as a chiral analogue of the 3×3 RH problem associated with the Pearcey differential equation used in [22, 23].

4.1 Third-order differential equation

In this section we prove the following result.

Proposition 4.1. *(Third order differential equation) Let W be quadratic as in (2.12). Then the function $p(x) := h_{0,n}(x)$ in (4.1) satisfies the third order differential equation*

$$x^2 p'''(x) + (2 - 2\nu)xp''(x) + (\alpha n^2 \tau^2 x + \nu^2 - \nu)p'(x) - (\tau^4 n^3 x + \tau^2 n^2 \nu \alpha)p(x) = 0. \quad (4.2)$$

We note that a similar differential equation appears in a Mehler-Heine type formula in [19]. To prove Proposition 4.1, we start with two lemmas.

Lemma 4.2. *The function $h_{l+1,n}$ in (2.3) can be expressed in terms of $h_{l,n}$ as*

$$(\tau n)^2 h_{l+1,n}(x) = xh''_{l,n}(x) - (\nu - 1)h'_{l,n}(x), \quad (4.3)$$

for any $l \in \mathbb{N} \cup \{0\}$.

Proof. This is obvious from (3.1) and (2.3). \square

Lemma 4.3. *Let W be quadratic as in (2.12). Then the functions $h_{l,n}$, $l = 0, 1, 2, 3$, satisfy the relation*

$$h_{3,n}(x) + 2\alpha h_{2,n}(x) + (\alpha^2 - (\nu + 2)n^{-1})h_{1,n}(x) - ((\nu + 1)n^{-1}\alpha + \tau^2 x)h_{0,n}(x) = 0. \quad (4.4)$$

Proof. We first assume that $\nu > 0$ and calculate

$$\begin{aligned} h_{3,n}(x) + 2\alpha h_{2,n}(x) + (\alpha^2 - n^{-1})h_{1,n}(x) &= \int_0^\infty y((y + \alpha)^2 - n^{-1})f_n(xy)e^{-nW(y)}dy \\ &= n^{-2} \int_0^\infty y f_n(xy) \frac{\partial^2}{\partial y^2} (e^{-nW(y)}) dy \\ &= n^{-2} \int_0^\infty x(xy f_n''(xy) + 2f_n'(xy)) e^{-nW(y)} dy, \end{aligned} \quad (4.5)$$

where the first step follows from the definition (2.3), the second step uses (2.12), and the third step is integration by parts. Similarly,

$$\begin{aligned} h_{1,n}(x) + \alpha h_{0,n}(x) &= \int_0^\infty (y + \alpha)f_n(xy)e^{-nW(y)}dy \\ &= -n^{-1} \int_0^\infty f_n(xy) \frac{\partial}{\partial y} (e^{-nW(y)}) dy \\ &= n^{-1} \int_0^\infty x f_n'(xy) e^{-nW(y)} dy. \end{aligned} \quad (4.6)$$

The lemma then follows from (3.1) by a little calculation. Finally, note that the assumption $\nu > 0$ was needed to make sure that the integration by parts in (4.5)–(4.6) creates no integrated terms. However, if $\nu \leq 0$, integrated terms are present, but cancel each other out so that (4.4) remains valid. \square

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. First observe that (3.4) and (2.12) yield the relations

$$xh'_{0,n}(x) = -h_{0,n}(x) + n\alpha h_{1,n}(x) + nh_{2,n}(x), \quad (4.7)$$

$$xh'_{1,n}(x) = -2h_{1,n}(x) + n\alpha h_{2,n}(x) + nh_{3,n}(x), \quad (4.8)$$

and so on. Multiplying (4.7) with αn^{-1} and (4.8) with n^{-1} , and adding them up, we get

$$h_{3,n}(x) + 2\alpha h_{2,n}(x) + \alpha^2 h_{1,n}(x) = n^{-1} (\alpha x h'_{0,n}(x) + \alpha h_{0,n}(x) + x h'_{1,n}(x) + 2h_{1,n}(x)).$$

Substituting this into (4.4), we find

$$xh'_{1,n}(x) - \nu h_{1,n}(x) + \alpha x h'_{0,n}(x) - (\nu\alpha + \tau^2 nx)h_{0,n}(x) = 0. \quad (4.9)$$

On account of (4.3) with $l = 0$ this yields

$$\begin{aligned} (\tau n)^2 h_{1,n}(x) &= xh''_{0,n}(x) - (\nu - 1)h'_{0,n}(x), \\ (\tau n)^2 h'_{1,n}(x) &= xh'''_{0,n}(x) - (\nu - 2)h''_{0,n}(x). \end{aligned}$$

Inserting this in (4.9) we get

$$\begin{aligned} x(\tau n)^{-2}(xh'''_{0,n}(x) - (\nu - 2)h''_{0,n}(x)) - \nu(\tau n)^{-2}(xh''_{0,n}(x) - (\nu - 1)h'_{0,n}(x)) \\ + \alpha x h'_{0,n}(x) - (\nu\alpha + \tau^2 nx)h_{0,n}(x) = 0, \end{aligned}$$

which with the notation $p(x) := h_{0,n}(x)$ is equivalent to (4.2). This proves Proposition 4.1. \square

Remark 4.4. A little calculation shows that for any solution $p(x)$ to (4.2) the transformed function

$$q(x) := x^{(1-4\nu)/3} p(x^2)$$

satisfies the differential equation

$$\begin{aligned} x^3 q'''(x) + \left(4\tau^2 n^2 \alpha x^3 - \frac{(2\nu+1)^2}{3} x \right) q'(x) \\ + \left(\frac{16\nu^3 + 60\nu^2 + 12\nu - 7}{27} - 8\tau^4 n^3 x^4 - \frac{4}{3} \tau^2 n^2 \alpha x^2 (2\nu+1) \right) q(x) = 0. \end{aligned}$$

For the special case $\nu = -1/2$ this reduces to the third order differential equation

$$q'''(x) + 4\tau^2 n^2 \alpha q'(x) = 8\tau^4 n^3 x q(x),$$

which appeared before in [23, eqn. (5.3)]. In that paper this ODE is solved using Pearcey integrals.

4.2 Three special solutions to (4.2)

We now construct three contour integral solutions p_0 , p_1 , and p_2 to the third-order differential equation (4.2). In the next section we will study the asymptotics of these functions as $z \rightarrow \infty$.

The function p_0 is defined as

$$p_0(z) := h_{0,n}(z) = \int_0^\infty f_n(zy) e^{-nW(y)} dy, \quad (4.10)$$

where we recall the definition of f_n (2.2). Then p_0 is analytic in $\mathbb{C} \setminus \mathbb{R}^-$.

The second solution p_1 to (4.2) is defined as

$$p_1(z) := \frac{i}{\pi} \int_{-\infty}^\infty (zy)^{\nu/2} K_\nu(2\tau n \sqrt{zy}) e^{-nW(y)} dy, \quad (4.11)$$

where

$$K_\nu(z) = \frac{1}{2} \pi \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \quad z \in \mathbb{C} \setminus (-\infty, 0] \quad (4.12)$$

is the modified Bessel function of the second kind. The right hand side of (4.12) is replaced by its limiting value if ν is an integer or zero. Then p_1 is analytic in $\mathbb{C} \setminus \mathbb{R}$.

Finally, the third solution to (4.2) is defined as

$$p_2(z) := \int_{-\infty}^0 f_n(zy) e^{-nW(y)} dy. \quad (4.13)$$

This function is analytic in $\mathbb{C} \setminus \mathbb{R}^+$.

The functions p_0, p_1, p_2 all satisfy the differential equation (4.2). For p_0 this follows directly from Proposition 4.1. The same proof works for p_2 . Finally, for p_1 we first observe that $(x)^{\nu/2} K_\nu(2\tau n \sqrt{x})$ satisfies the same differential equation (3.1) as $f_n(x)$ since also K_ν solves the modified Bessel equation (3.2). This then allows the proof of Proposition 4.1 to be applied word for word.

Lemma 4.5. *The functions p_0, p_1, p_2 satisfy the jump relations*

$$\begin{aligned} p_{0,+}(x) &= e^{2\nu\pi i} p_{0,-}(x), & x \in \mathbb{R}^-, \\ p_{1,+}(x) &= p_{1,-}(x) - e^{-\nu\pi i} p_{2,-}(x), & x \in \mathbb{R}^+, \\ p_{1,+}(x) &= p_{1,-}(x) + e^{\nu\pi i} p_{0,-}(x), & x \in \mathbb{R}^-, \\ p_{2,+}(x) &= e^{-2\nu\pi i} p_{2,-}(x), & x \in \mathbb{R}^+, \end{aligned}$$

where the subscripts $+$ and $-$ denote the boundary values obtained from the upper or lower half plane of \mathbb{C} , respectively.

Proof. This follows from the definitions (4.10), (4.11), and (4.13); the connection formula (4.12); and the fact that $I_{\nu+}(x) = I_{\nu-}(x)e^{2\nu\pi i}$ for $x < 0$ which is immediate from (1.5). \square

4.3 Asymptotics for the solutions to (4.2)

Our next goal is to find the large z asymptotics of p_0 , p_1 , and p_2 . In analogy with [23] we first introduce three functions $\theta_j(z)$, $j = 1, 2, 3$, that will appear in these asymptotics.

First we define the constants $x^*(\alpha)$ and $y^*(\alpha)$ that depend on α and τ . We put

$$x^*(\alpha) = \begin{cases} 0, & \alpha \geq 0, \\ -\frac{4}{\tau^2}(\frac{\alpha}{3})^3, & \alpha < 0, \end{cases} \quad y^*(\alpha) = -x^*(-\alpha). \quad (4.14)$$

So if $\alpha > 0$ we have that $x^*(\alpha) = 0$ and $y^*(\alpha) < 0$ whereas in the case $\alpha \leq 0$ it holds that $x^*(\alpha) \geq 0$ and $y^*(\alpha) = 0$.

Lemma 4.6. *For every $\alpha \in \mathbb{R}$ and $\tau > 0$ there exist analytic functions $\theta_j : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$, $j = 1, 2, 3$, with the following properties.*

- (a) *The jumps of the functions θ_j are taken together in terms of the jumps for the diagonal matrix*

$$\Theta(z) = \text{diag}(\theta_1(z), \theta_2(z), \theta_3(z)), \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (4.15)$$

We have

$$\begin{cases} \Theta_+(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Theta_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & x > x^*(\alpha), \\ \Theta_+(x) = \Theta_-(x), & y^*(\alpha) < x < x^*(\alpha), \\ \Theta_+(x) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Theta_-(x) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & x < y^*(\alpha), \end{cases}$$

with $x^(\alpha)$ and $y^*(\alpha)$ as in (4.14).*

- (b) *We have as $z \rightarrow \infty$ within \mathbb{C}^+*

$$\begin{aligned} \theta_j(z) = & \frac{3}{2}\omega^{j-1}\tau^{4/3}z^{2/3} - \alpha\omega^{4-j}\tau^{2/3}z^{1/3} + \frac{\alpha^2}{3} \\ & - \frac{\alpha^3}{27}\omega^{j-1}\tau^{-2/3}z^{-1/3} + D\omega^{4-j}\tau^{-4/3}z^{-2/3} + \mathcal{O}(z^{-1}), \end{aligned}$$

for a constant $D \in \mathbb{R}$ and $j = 1, 2, 3$. Here $\omega := \exp(2\pi i/3)$. The behavior in \mathbb{C}^- follows from the relation $\theta_j(\bar{z}) = \overline{\theta_j(z)}$, $j = 1, 2, 3$.

Proof. Define

$$\theta_j(z) = 2\theta_j^{\text{NC}}(\sqrt{z}), \quad j = 1, 2, 3, \quad (4.16)$$

where the θ_j^{NC} refer to the θ -functions used in [23] with potentials given in (2.19). From (2.5) and (2.8) in [23], it is readily seen that

$$\theta_j^{\text{NC}}(-z) = \theta_j^{\text{NC}}(z). \quad (4.17)$$

This, together with [23, Corollary 2.2 and Lemma 2.4], implies our lemma. \square

We are now ready to investigate the large z asymptotics of p_0 , p_1 , and p_2 . Let us first give a heuristic argument.

Lemma 4.7. *For any solution $p(z)$ to (4.2), there exist $j \in \{0, 1, 2\}$ and $C \in \mathbb{C} \setminus \{0\}$ such that*

$$p(z) = Cz^{\frac{2\nu-1}{3}} e^{n\theta_{j+1}(z)} \left(1 - \frac{\alpha\nu}{3\tau^{2/3}n} \omega^j z^{-1/3} + \tilde{D} \omega^{2j} z^{-2/3} + \mathcal{O}(z^{-1}) \right), \quad (4.18)$$

as $z \rightarrow \infty$ in closed sectors of the first quadrant $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$. Here $\tilde{D} \in \mathbb{R}$ is a constant, and θ_j is defined in Lemma 4.6.

Proof. Substituting $p(z) = e^{nF(z)}$ in (4.2) we get the following nonlinear ODE for $f(z) := F'(z)$

$$\begin{aligned} (z^2 f^3(z) - \tau^4 z + \alpha \tau^2 z f(z)) + n^{-1} (3z^2 f(z) f'(z) - 2(\nu - 1) z f^2(z) - \nu \tau^2 \alpha) \\ + n^{-2} (z^2 f''(z) - 2(\nu - 1) z f'(z) + \nu(\nu - 1) f(z)) = 0. \end{aligned}$$

This ODE has solutions f with expansions

$$\begin{aligned} \tau^{-2} f(z/\tau^2) = \omega^j z^{-1/3} - \frac{\alpha}{3} \omega^{2j} z^{-2/3} + \frac{2\nu-1}{3n} z^{-1} + \frac{\alpha(9\nu+n\alpha^2)}{81n} \omega^j z^{-4/3} \\ + \frac{\omega^{2j}}{243n^2} (27\nu(\nu+1) - 9 + 9(\nu+1)\alpha^2 n + \alpha^4 n^2) z^{-5/3} + \mathcal{O}(z^{-2}), \end{aligned}$$

as $z \rightarrow \infty$, for any $j \in \{0, 1, 2\}$. If $\operatorname{Im} z > 0$, after integration this yields

$$F(z) = \theta_{j+1}(z) + \frac{2\nu-1}{3n} \log z + \frac{1}{n} \log C - \frac{\alpha\nu}{3\tau^{2/3}n} \omega^j z^{-1/3} + D' \omega^{2j} z^{-2/3} + \mathcal{O}(z^{-1}),$$

where C is the integration constant and $D' \in \mathbb{R}$. Then $p(z) = e^{nF(z)}$ satisfies (4.18). \square

Next we specialize Lemma 4.7 to the functions p_0 , p_1 , and p_2 .

Lemma 4.8. *For each $j = 0, 1, 2$, we have*

$$p_j(z) = \frac{C_j}{\sqrt{3}n} \tau^{\frac{\nu-2}{3}} e^{\alpha^2 n/3} \omega^{2j} z^{\frac{2\nu-1}{3}} e^{n\theta_{j+1}(z)} \left(1 - \frac{\alpha\nu}{3\tau^{2/3}n} \omega^j z^{-1/3} + \tilde{D} \omega^{2j} z^{-2/3} + \mathcal{O}(z^{-1}) \right),$$

as $z \rightarrow \infty$ in closed sectors of the upper half plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, where

$$C_0 = 1, \quad C_1 = -ie^{-\frac{\pi i}{6}(2\nu+1)}, \quad C_2 = e^{-\frac{\pi i}{3}(2\nu+1)}, \quad (4.19)$$

and \tilde{D} is the constant in (4.18). In the lower half plane we have the same expansions but with ω replaced by ω^{-1} and C_0, C_1, C_2 by $\overline{C_0}, -\overline{C_1}, \overline{C_2}$ respectively.

Proof. First assume $\tau = n = 1$. Consider the ODE

$$zq'''(z) + \nu q''(z) + \alpha q'(z) - q(z) = 0. \quad (4.20)$$

As observed in [19], for any solution $q(z)$ of (4.20)

$$p(z) = z^\nu q''(z)$$

solves (4.2). In [32] solutions to (4.20) are studied. The authors consider four solutions of the form

$$q_j(z) = \int_{\Gamma_j} t^{\nu-3} e^{\frac{1}{2t^2} - \frac{\alpha}{t} + zt} dt, \quad j = 1, 2, 3, 4, \quad (4.21)$$

where the contours Γ_j are shown in Figure 4.

The branch cuts for $t^{\nu-3}$ are specified as follows: for $j = 1$ we take $-\pi/2 < \arg t < \pi/2$, for $j = 2$ we have $\pi/2 < \arg t < 3\pi/2$, for $j = 3$ we choose $0 < \arg t < \pi$, and for $j = 4$ we

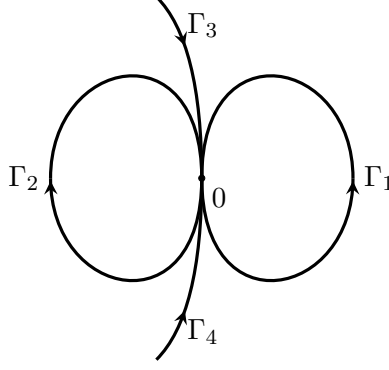


Figure 4: The contours of integration Γ_j , $j = 1, 2, 3, 4$, used in the definitions (4.21) of the functions q_j .

put $-\pi < \arg t < 0$. Note that our definition differs from the one in [32] by a multiplicative constant.

We have the following relations

$$q_2(z) = q_3(z) - e^{2\pi\nu i} q_4(z), \quad \text{for } \operatorname{Re} z > 0, \quad (4.22)$$

$$q_1(z) = q_3(z) - q_4(z), \quad \text{for } \operatorname{Re} z < 0. \quad (4.23)$$

It is then clear that the functions

$$z^\nu q_j''(z) = z^\nu \int_{\Gamma_j} t^{\nu-1} e^{\frac{1}{2t^2} - \frac{\alpha}{t} + zt} dt, \quad j = 1, 2, 3, 4,$$

generate the three dimensional solution space of (4.2). Moreover [32] provides detailed asymptotics of these functions, e.g. as $z \rightarrow \infty$ such that $0 < \arg z < \pi/4$ we have

$$z^\nu q_1''(z) = \sqrt{2\pi} e^{-\alpha^2/6} \frac{i}{\sqrt{3}} z^{\frac{2\nu-1}{3}} e^{\frac{3}{2}z^{2/3} - \alpha z^{1/3}} \left(1 + \mathcal{O}(z^{-1/3})\right), \quad (4.24)$$

$$z^\nu q_3''(z) = \sqrt{2\pi} e^{-\alpha^2/6} e^{\frac{2\nu\pi i}{3}} \frac{i}{\sqrt{3}} \omega z^{\frac{2\nu-1}{3}} e^{\frac{3}{2}\omega z^{2/3} - \alpha \omega z^{1/3}} \left(1 + \mathcal{O}(z^{-1/3})\right), \quad (4.25)$$

$$z^\nu q_4''(z) = -\sqrt{2\pi} e^{-\alpha^2/6} e^{-\frac{2\nu\pi i}{3}} \frac{i}{\sqrt{3}} \omega^2 z^{\frac{2\nu-1}{3}} e^{\frac{3}{2}\omega^2 z^{2/3} - \alpha \omega^2 z^{1/3}} \left(1 + \mathcal{O}(z^{-1/3})\right). \quad (4.26)$$

The asymptotic behavior of $z^\nu q_2''(z)$ follows using the connection formula (4.22). The idea is now to express p_0 , p_1 , and p_2 as linear combinations of the functions $z^\nu q_j''(z)$ to obtain the asymptotic behavior. This is done in Lemma 4.9 below. It leads to the following results

$$p_0(z) = \frac{1}{\sqrt{3}} e^{\alpha^2/3} z^{\frac{2\nu-1}{3}} e^{\frac{3}{2}z^{2/3} - \alpha z^{1/3}} \left(1 + \mathcal{O}(z^{-1/3})\right), \quad (4.27)$$

$$p_1(z) = -\frac{i}{\sqrt{3}} e^{\alpha^2/3} e^{-\frac{\pi i}{6}(2\nu+1)} \omega^2 z^{\frac{2\nu-1}{3}} e^{\frac{3}{2}\omega z^{2/3} - \alpha \omega z^{1/3}} \left(1 + \mathcal{O}(z^{-1/3})\right), \quad (4.28)$$

$$p_2(z) = \frac{1}{\sqrt{3}} e^{\alpha^2/3} e^{-\frac{\pi i}{3}(2\nu+1)} \omega z^{\frac{2\nu-1}{3}} e^{\frac{3}{2}\omega^2 z^{2/3} - \alpha \omega^2 z^{1/3}} \left(1 + \mathcal{O}(z^{-1/3})\right), \quad (4.29)$$

as $z \rightarrow \infty$ such that $0 < \arg z < \pi/4$. This proves the lemma for $\tau = n = 1$ and in the sector $0 < \arg z < \pi/4$. The results in the remaining sectors are proved similarly. The lemma for general values of τ and n follows by the rescaling

$$p_j^{\alpha, n, \tau, \nu}(z) = \frac{1}{(\tau n)^\nu \sqrt{n}} p_j^{\sqrt{n}\alpha, n=\tau=1, \nu} \left(n^{\frac{3}{2}} \tau^2 z\right), \quad j = 0, 1, 2. \quad (4.30)$$

This equality follows from the definitions (4.10)–(4.13) changing variables in the integrals. Note that this rescaling is consistent with the ODE (4.2). \square

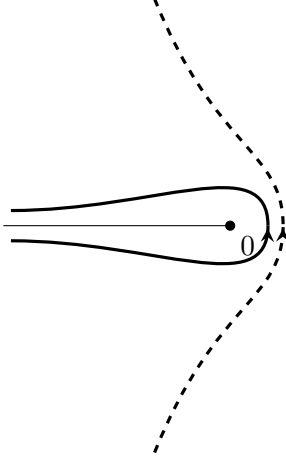


Figure 5: Hankel contour (solid) used for integration in (4.34) and its deformation (dashed).

Lemma 4.9. *Assuming $\tau = n = 1$, the following relations between p_j , $j = 0, 1, 2$, and q_j , $j = 1, 2, 3$, hold*

$$p_0(z) = \frac{1}{\sqrt{2\pi i}} e^{\alpha^2/2} z^\nu q_1''(z), \quad z \in \mathbb{C} \setminus \mathbb{R}^-, \quad (4.31)$$

$$p_1(z) = \begin{cases} \frac{1}{\sqrt{2\pi i}} e^{-\nu\pi i} e^{\alpha^2/2} z^\nu q_3''(z), & \text{Im } z > 0, \\ \frac{1}{\sqrt{2\pi i}} e^{\nu\pi i} e^{\alpha^2/2} z^\nu q_4''(z), & \text{Im } z < 0, \end{cases} \quad (4.32)$$

$$p_2(z) = -\frac{1}{\sqrt{2\pi i}} e^{-\nu\pi i} e^{\alpha^2/2} (-z)^\nu q_2''(z) \quad z \in \mathbb{C} \setminus \mathbb{R}^+. \quad (4.33)$$

Proof. We will prove (4.31)–(4.33) for $\nu \in (-1, \infty) \setminus \mathbb{Z}$. When $\nu \in \mathbb{Z}$ the equalities still hold true by continuity in ν .

We start with the proof of (4.31) using the convergent series representations

$$p_0(z) = z^\nu \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(k + \nu + 1)} \int_0^\infty y^{k+\nu} e^{-W(y)} dy,$$

$$z^\nu q_1''(z) = z^\nu \sum_{k=0}^{\infty} \frac{z^k}{k!} \int_{\Gamma_1} t^{\nu-1+k} e^{\frac{1}{2t^2} - \frac{\alpha}{t}} dt.$$

Recall the contour integral formula for the reciprocal of the Gamma function [1]

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int e^{t-t^z} dt, \quad (4.34)$$

where the integration is over a Hankel contour as shown in Figure 5. The Cauchy theorem allows us to deform this contour of integration into a contour coming from infinity under the angle $-\phi$, avoiding the negative real line, and tending to infinity under the angle ϕ , for $\pi/2 < \phi < 3\pi/4$, see Figure 5. Then

$$\frac{1}{\Gamma(\nu + k + 1)} \int_0^\infty y^{\nu+k} e^{-\frac{y^2}{2} - \alpha y} dy = \frac{1}{2\pi i} \int_0^\infty y^{\nu+k} \int t^{-\nu-1-k} e^{-\frac{y^2}{2} - \alpha y + t} dt dy,$$

where the inner integral is taken over the deformed Hankel contour. In the inner integral (thus, for fixed $y > 0$) we change variables $t = y/s$

$$\frac{1}{\Gamma(\nu + k + 1)} \int_0^\infty y^{\nu+k} e^{-\frac{y^2}{2} - \alpha y} dy = \frac{1}{2\pi i} \int_0^\infty \int_{\Gamma_1^*} s^{\nu-1+k} e^{-\frac{y^2}{2} - \alpha y + \frac{y}{s}} ds dy.$$

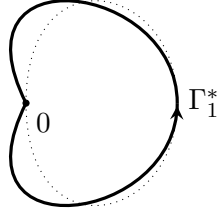


Figure 6: Contour of integration Γ_1^* (solid) used in the proof of Lemma 4.9 and the contour Γ_1 (dotted).

The new contour Γ_1^* of integration emerges from zero under the angle $-\phi$ and ends in zero under the angle ϕ , where $\pi/2 < \phi < 3\pi/4$. It has counterclockwise orientation, see Figure 6. Using the Fubini theorem we can change the order of integration to become

$$\begin{aligned} \frac{1}{\Gamma(\nu+k+1)} \int_0^\infty y^{\nu+k} e^{-\frac{y^2}{2}-\alpha y} dy &= \frac{1}{2\pi i} \int_{\Gamma_1^*} s^{\nu-1+k} \int_{-\infty}^\infty e^{-\frac{y^2}{2}-\alpha y+\frac{y}{s}} dy ds \\ &\quad - \frac{1}{2\pi i} \int_{\Gamma_1} s^{\nu-1+k} \int_{-\infty}^0 e^{-\frac{y^2}{2}-\alpha y+\frac{y}{s}} dy ds. \end{aligned} \quad (4.35)$$

We complete the square in the first term of the right-hand side of (4.35) and evaluate the Gaussian integral. Then we can deform the contour Γ_1^* into Γ_1 . Hence the first term of the right-hand side of (4.35) equals

$$\frac{1}{\sqrt{2\pi i}} e^{\frac{\alpha^2}{2}} \int_{\Gamma_1} t^{\nu-1+k} e^{\frac{1}{2t^2}-\frac{\alpha}{t}} dt.$$

The second term in the right-hand side of (4.35) vanishes, since we can contract the contour of integration thanks to Lemma 4.10. This completes the proof of (4.31).

Next we prove (4.33). We claim that the following chain of equalities holds

$$p_2(z) = \tilde{p}_0(-z) = \frac{1}{\sqrt{2\pi i}} e^{\alpha^2/2} (-z)^\nu \tilde{q}_1''(-z) = \frac{1}{\sqrt{2\pi i}} e^{\pi i(1-\nu)} e^{\alpha^2/2} (-z)^\nu q_2''(z), \quad (4.36)$$

for $z \in \mathbb{C} \setminus \mathbb{R}^+$. Here, a tilde means that the parameter α on which the quantity depends has to be replaced by $-\alpha$. Then (4.33) is immediate from the claim. We now discuss the equalities in (4.36). The first one follows from (4.10) and (4.13). The second equality relies on (4.31). The last equality is a consequence of (4.21) for $j = 1, 2$, where one has to take special care about the position of the branch cut.

Finally we prove (4.32). Using (4.10)–(4.13) and (1.5) we find

$$-2i \sin(\nu\pi) p_1(z) + p_0(z) + p_2(z) = \int_{-\infty}^\infty \sum_{k=0}^\infty \frac{1}{k! \Gamma(k-\nu+1)} (zy)^k e^{-y^2/2-\alpha y} dy$$

This function belongs to the one-parameter family of entire solutions of the ODE (4.2). Therefore it is characterized by its value at zero

$$\frac{\sqrt{2\pi}}{\Gamma(1-\nu)} e^{\alpha^2/2}.$$

The idea is now to build an entire function as a linear combination of the $z^\nu q_j''(z)$, $j = 1, 2, 3, 4$ and find its value at zero to obtain the desired equality.

When $x > 0$ we have

$$x^\nu (q_4''(x) + q_1''(x) - q_3''(x)) = \int s^{\nu-1} e^{\frac{x^2}{2s^2} - \alpha \frac{x}{s} + s} ds, \quad (4.37)$$

where we used (4.21) and made the change of variables $s = xt$. The integral is taken over the dashed contour in Figure 5. Clearly the right-hand side of this expression is entire in x . Therefore the equality is valid in the right half plane $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. The analytic continuation to the left half plane takes the form

$$-z^\nu (e^{2\pi i \nu} q_4''(z) + q_2''(z) - q_3''(z)) = \int s^{\nu-1} e^{\frac{z^2}{2s^2} - \alpha \frac{z}{s} + s} ds, \quad \operatorname{Re} z < 0,$$

where the integration is over the same deformed Hankel contour and the argument of z is chosen in the interval $(-\pi/2, -\pi/2)$.

We can use the right-hand side of (4.37) to determine its value at zero

$$\frac{2\pi i}{\Gamma(1-\nu)},$$

where we also used (4.34). Hence we find the equality

$$-2i \sin(\nu\pi) p_1(z) + p_0(z) + p_2(z) = \frac{1}{\sqrt{2\pi i}} e^{\frac{\alpha^2}{2}} z^\nu (q_4''(z) + q_1''(z) - q_3''(z)), \quad \operatorname{Re} z > 0.$$

Now apply (4.31), (4.33) (assuming $\operatorname{Im} z > 0$), and (4.22) to obtain (4.32). In the other quadrants similar relations can be found. \square

Lemma 4.10. *The function*

$$f_\alpha(s) = \int_{-\infty}^0 e^{-\frac{y^2}{2} - \alpha y + \frac{y}{s}} dy$$

is analytic in $\mathbb{C} \setminus \{0\}$ and satisfies $f_\alpha(s) = \mathcal{O}(s)$ as $s \rightarrow 0$ uniformly within the sector $|\arg s| \leq 3\pi/4 - \delta$, for any small $\delta > 0$.

Proof. We rewrite $f_\alpha(s)$ as

$$f_\alpha(s) = e^{\frac{t^2}{2}} \sqrt{\frac{\pi}{2}} \operatorname{erfc}\left(\frac{t}{\sqrt{2}}\right), \quad t = \frac{1}{s} - \alpha,$$

where the complementary error function is defined as

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-y^2} dy.$$

This function has uniform asymptotics

$$\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}z} (1 + \mathcal{O}(z^{-2})),$$

as $z \rightarrow \infty$ such that $|\arg z| \leq 3\pi/4 - \delta$, for a small $\delta > 0$, see [1]. Then $f_\alpha(s)$ has the asymptotics

$$f_\alpha(s) = \frac{s}{1 - \alpha s} \left(1 + \mathcal{O}\left(\frac{s^2}{(1 - \alpha s)^2}\right)\right) = \mathcal{O}(s),$$

as $s \rightarrow 0$ such that $|\arg s| \leq 3\pi/4 - \delta$. This proves the lemma. \square

4.4 Wronskian matrix

From the three special solutions p_j , $j = 1, 2, 3$, introduced in the previous section we construct the Wronskian matrix

$$W_n(z) = \begin{pmatrix} p_0(z) & p_1(z) & p_2(z) \\ p'_0(z) & p'_1(z) & p'_2(z) \\ p''_0(z) & p''_1(z) & p''_2(z) \end{pmatrix} \times \text{diag}(1, 1, e^{\pm \nu \pi i}), \quad \pm \text{Im } z > 0. \quad (4.38)$$

This Wronskian matrix will be used in the first transformation of the RH problem for $Y(z)$, see Section 5.1.

Proposition 4.11. *The Wronskian matrix W_n satisfies the following RH problem.*

- (1) $W_n(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.
- (2) W_n has jumps on \mathbb{R}^+ and \mathbb{R}^- given by

$$W_{n,+}(x) = W_{n,-}(x) \times \begin{cases} \text{diag}\left(1, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\right), & x \in \mathbb{R}^+, \\ \text{diag}\left(\begin{pmatrix} e^{2\nu\pi i} & e^{\nu\pi i} \\ 0 & 1 \end{pmatrix}, e^{2\nu\pi i}\right), & x \in \mathbb{R}^-. \end{cases} \quad (4.39)$$

- (3) As $z \rightarrow \infty$, we have that

$$\begin{aligned} W_n(z) &= 3^{-\frac{1}{2}\tau} \tau^{\frac{\nu+2}{3}} e^{\frac{\alpha^2}{3n}} z^{\frac{2\nu-2}{3}} \text{diag}(n^{-1}\tau^{-4/3}z^{1/3}, 1, n\tau^{4/3}z^{-1/3}) \\ &\times \left(I + \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix} z^{-1/3} + \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix} z^{-2/3} + O(z^{-1}) \right) \begin{pmatrix} 1 & \omega^2 & \omega \\ 1 & 1 & 1 \\ 1 & \omega & \omega^2 \end{pmatrix} \\ &\times \text{diag}(1, -\sigma^{-1}, \sigma) e^{n\Theta(z)}, \end{aligned} \quad (4.40)$$

as $z \rightarrow \infty$ in closed sectors of the upper half plane $\{z \in \mathbb{C} \mid \text{Im } z > 0\}$. Here the $*$ denote certain constant real entries and

$$\sigma := e^{\frac{\pi i}{3}(\nu-1)}. \quad (4.41)$$

We have the same expression as $z \rightarrow \infty$ in closed sectors of the lower half plane, but then with ω replaced by ω^{-1} and $\text{diag}(1, -\sigma^{-1}, \sigma)$ by $\text{diag}(1, \sigma, \sigma^{-1})$.

Proof. This follows from Lemmas 4.5 and 4.8. \square

Note that $W_n(z)$ is not uniquely determined from the above RH problem, since we did not specify the behavior near the origin $z = 0$. This will be done in (5.9) below.

Lemma 4.12. *The asymptotics of W_n can be rewritten as*

$$\begin{aligned} W_n(z) &= -i\tau^{\frac{\nu+2}{3}} e^{\frac{\alpha^2}{3n}} z^{\frac{2\nu-2}{3}} \left[\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} + O(z^{-1}) \right] \text{diag}(n^{-1}\tau^{-4/3}z^{1/3}, 1, n\tau^{4/3}z^{-1/3}) \\ &\times A_{\pm} \text{diag}(1, \sigma^{\mp 1}, \sigma^{\pm 1}) e^{n\Theta(z)}, \end{aligned} \quad (4.42)$$

as $z \rightarrow \infty$ in closed sectors of the half plane $\{z \in \mathbb{C} \mid \pm \text{Im } z > 0\}$, with σ as in (4.41) and

$$A_+ := \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & -\omega^2 & \omega \\ 1 & -1 & 1 \\ 1 & -\omega & \omega^2 \end{pmatrix}, \quad A_- := \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 1 \\ 1 & \omega^2 & \omega \end{pmatrix}. \quad (4.43)$$

The prefactors in (4.43) are chosen such that both matrices have determinant 1.

Remark 4.13. It is easily seen from (4.2) and (4.38) that the determinant of W_n satisfies the following linear differential equation

$$z(\det W_n)'(z) = (2\nu - 2)\det W_n(z), \quad z \in \mathbb{C} \setminus \mathbb{R}^-. \quad (4.44)$$

Hence, we have for a certain nonzero constant K that

$$\det W_n(z) = Kz^{2\nu-2}. \quad (4.45)$$

5 Steepest descent analysis for $Y(z)$ with quadratic potential W : regular cases

In this section we will perform a Deift/Zhou steepest descent analysis for $Y(z)$ in the situation that W is quadratic (2.12) and the triplet (V, W, τ) is regular in the sense of Definition 2.5. This analysis consists of a series of invertible transformations and results in the proof of Theorem 2.6.

5.1 First transformation $Y \mapsto X$

The idea behind the first transformation $Y \mapsto X$ is inspired by [22]. We will multiply the RH matrix Y on the right with the inverse transpose of the Wronskian matrix W_n . Recall from (4.38) that W_n is constructed from the solutions to the third-order differential equation satisfied by $h_{0,n}$.

In the analysis we will work with the alternative system of weight functions in (2.6) rather than those in (2.5). Thanks to (2.7), (4.3), and (4.10) these weight functions are

$$\begin{cases} w_{0,n}(z) = e^{-nV(z)}p_0(z), \\ w_{1,n}(z) = e^{-nV(z)}zp_0'(z), \\ w_{2,n}(z) = e^{-nV(z)}(zp_0''(z) + (1-\nu)p_0'(z)). \end{cases} \quad (5.1)$$

Note that $w_{1,n}$ and $w_{2,n}$ have been interchanged with respect to (2.7). The labeling (5.1) will be more convenient for us. Defining

$$D(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 1-\nu \\ 0 & 0 & z \end{pmatrix}, \quad (5.2)$$

(5.1) leads to the equality

$$e^{-nV(z)}W_n^T(z)D(z) = \begin{pmatrix} w_{0,n}(z) & w_{1,n}(z) & w_{2,n}(z) \\ * & * & * \\ * & * & * \end{pmatrix}, \quad (5.3)$$

where $*$ denotes unimportant matrix entries. This identity motivates the transformation $Y \mapsto X$ given below.

Definition 5.1. We define X by

$$X(z) = z^{\nu/2}P_n^{-1}Y(z)\operatorname{diag}\left(1, D^{-1}(z)W_n^{-T}(z)e^{n\Theta(z)}\right), \quad (5.4)$$

for $z \in \mathbb{C} \setminus \mathbb{R}$, where $\Theta(z)$ is defined in (4.15) and $D(z)$ in (5.2). P_n is an invertible matrix to be specified below (5.7).

We claim that $X(z)$ is the unique solution to RH problem 5.2 below. Note that the jump on the positive real axis is simplified at the cost of creating a jump on the negative real axis.

RH problem 5.2. *The 4×4 matrix-valued function $X : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{4 \times 4}$ defined in (5.4) satisfies the following conditions.*

(1) $X(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.

(2) We have

$$X_+(x) = X_-(x)J_X(x), \quad x \in \mathbb{R},$$

where the jump matrices J_X are given by

$$J_X(x) = \text{diag}\left(1, e^{-n\Theta_-(x)}\right) \begin{pmatrix} 1 & e^{-nV(x)} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{diag}\left(1, e^{n\Theta_+(x)}\right),$$

if $x > 0$,

$$J_X(x) = \text{diag}\left(1, e^{-n\Theta_-(x)}\right) \begin{pmatrix} e^{\nu\pi i} & 0 & 0 & 0 \\ 0 & e^{-\nu\pi i} & 0 & 0 \\ 0 & -1 & e^{\nu\pi i} & 0 \\ 0 & 0 & 0 & e^{-\nu\pi i} \end{pmatrix} \text{diag}\left(1, e^{n\Theta_+(x)}\right),$$

if $x < 0$. See (4.15) for the definition of Θ .

(3) We have as $z \rightarrow \infty$ with $\pm \text{Im } z > 0$,

$$X(z) = [I + \mathcal{O}(z^{-1})] \text{diag}\left(z^n, z^{-\frac{n-1}{3}}, z^{-\frac{n+1}{3}}, z^{-\frac{n}{3}}\right) \text{diag}\left(z^{\nu/2}, z^{-\nu/6} A_{\pm}^{-T}\right) \times \text{diag}\left(1, 1, \sigma^{\pm 1}, \sigma^{\mp 1}\right), \quad (5.5)$$

where σ and A_{\pm} are defined in (4.41) and (4.43), respectively.

(4) As $z \rightarrow 0$, $z \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\begin{cases} X(z) \text{diag}(|z|^{-\nu/2}, |z|^{\nu/2}, |z|^{-\nu/2}, |z|^{\nu/2}) = \mathcal{O}(1), & \text{if } \nu > 0, \\ X(z) \text{diag}(1, (\log |z|)^{-1}, 1, (\log |z|)^{-1}) = \mathcal{O}(1), & \text{if } \nu = 0, \\ X(z) = \mathcal{O}(|z|^{\nu/2}), \quad X^{-1}(z) = \mathcal{O}(|z|^{\nu/2}), & \text{if } -1 < \nu < 0. \end{cases} \quad (5.6)$$

Proof. The jumps of X follow directly from (4.39), (5.3) and the definitions. Next we check the asymptotics for $z \rightarrow \infty$. From (2.9) and (5.2) we obtain

$$Y(z) \text{diag}\left(1, D^{-1}(z)\right) = (I + \mathcal{O}(z^{-1})) \text{diag}\left(z^n, z^{-n/3}, z^{-n/3-1}, z^{-n/3-1}\right),$$

as $z \rightarrow \infty$. Moreover, Lemma 4.12 yields

$$z^{\nu/2} W_n^{-T}(z) e^{n\Theta(z)} = i\tau^{-\frac{\nu+2}{3}} e^{-\frac{\alpha^2}{3n}} z^{\frac{2}{3}-\frac{\nu}{6}} \left[\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} + \mathcal{O}(z^{-1}) \right] \times \text{diag}\left(\tau^{4/3} n z^{-1/3}, 1, \tau^{-4/3} n^{-1} z^{1/3}\right) A_{\pm}^{-T} \text{diag}\left(1, \sigma^{\pm 1}, \sigma^{\mp 1}\right),$$

as $z \rightarrow \infty$ in a sector of $\pm \text{Im } z > 0$. Combining this we get

$$\begin{aligned}
& z^{\nu/2} Y(z) \text{diag} \left(1, D^{-1}(z) W_n^{-T}(z) e^{n\Theta(z)} \right) \\
&= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\tau^{\frac{2-\nu}{3}} e^{-\frac{\alpha^2}{3n}n} & * & * \\ 0 & 0 & i\tau^{-\frac{2+\nu}{3}} e^{-\frac{\alpha^2}{3n}n} & 0 \\ 0 & 0 & * & i\tau^{-\frac{6+\nu}{3}} e^{-\frac{\alpha^2}{3n}n^{-1}} \end{pmatrix} + \mathcal{O}(z^{-1}) \right] \\
&\times \text{diag} \left(z^n, z^{-\frac{n-1}{3}}, z^{-\frac{n+1}{3}}, z^{-\frac{n}{3}} \right) \text{diag} \left(z^{\nu/2}, z^{-\nu/6} A_{\pm}^{-T} \right) \text{diag} (1, 1, \sigma^{\pm 1}, \sigma^{\mp 1}), \quad (5.7)
\end{aligned}$$

as $z \rightarrow \infty$ with $\pm \text{Im } z > 0$, for certain constants $*$. Finally, we define P_n to be the constant matrix in the square bracket in (5.7), which yields (5.5).

Next we show the behavior near the origin in (5.6). Recall the three weight functions $e^{-nV(x)} h_{0,n}(x)$, $e^{-nV(x)} x h'_{0,n}(x)$, and $e^{-nV(x)} (x h''_{0,n}(x) + (1-\nu) h'_{0,n}(x))$ in the RH problem for Y . It follows from the explicit form of Y that

$$\begin{cases} Y(z) = \mathcal{O}(1), & \text{if } \nu > 0, \\ Y(z) \text{diag}(1, (\log |z|)^{-1}, 1, (\log |z|)^{-1}) = \mathcal{O}(1), & \text{if } \nu = 0, \\ Y(z) \text{diag}(1, |z|^{-\nu}, |z|^{-\nu}, |z|^{-\nu}) = \mathcal{O}(1), & \text{if } -1 < \nu < 0, \\ Y(z)^{-T} \text{diag}(|z|^{-\nu}, 1, 1, 1) = \mathcal{O}(1), & \text{if } -1 < \nu < 0, \end{cases} \quad (5.8)$$

as $z \rightarrow 0$. On the other hand, we see from the definitions of p_i , $i = 0, 1, 2$, that there exist some constants A, B, C, D that depend on ν such that

$$\begin{aligned}
p_0(z) &= Az^\nu + \mathcal{O}(|z|^{\nu+1}), \\
p_1(z) &= \begin{cases} Bz^\nu + \mathcal{O}(|z|^{\nu+1}) + C + \mathcal{O}(z), & \text{if } \nu \neq 0, \\ B + \mathcal{O}(z) + \log z (C + \mathcal{O}(z)), & \text{if } \nu = 0, \end{cases} \\
p_2(z) &= Dz^\nu + \mathcal{O}(|z|^{\nu+1}), \end{aligned} \quad (5.9)$$

as $z \rightarrow 0$. This, together with (5.2) and (4.38) gives

$$D^T(z) W_n(z) = \begin{pmatrix} Az^\nu + \mathcal{O}(|z|^{\nu+1}) & \mathcal{O}(|z|^\nu) + \mathcal{O}(1) + \mathcal{O}(\log |z|) & Dz^\nu + \mathcal{O}(|z|^{\nu+1}) \\ \nu Az^\nu + \mathcal{O}(|z|^{\nu+1}) & \mathcal{O}(|z|^\nu) + \mathcal{O}(z) & \nu Dz^\nu + \mathcal{O}(|z|^{\nu+1}) \\ \mathcal{O}(|z|^\nu) & \mathcal{O}(|z|^\nu) + \mathcal{O}(1) + \mathcal{O}(\log |z|) & \mathcal{O}(|z|^\nu) \end{pmatrix},$$

as $z \rightarrow 0$, where we understand that the $\mathcal{O}(\log |z|)$ terms are absent if $\nu \neq 0$. Also in case $\nu = 0$ the $(1, 2)$ and the $(3, 2)$ entry have to be replaced by $\mathcal{O}(z)$. Note that the terms of order $\mathcal{O}(|z|^{\nu-1})$ in the last row all cancel.

Using the cofactor formula for the inverse transpose and the fact that

$$\det(D^T(z) W_n(z)) = K z^{2\nu},$$

see (4.45), we find

$$\begin{aligned}
& D^{-1}(z) W_n^{-T}(z) \\
&= \frac{1}{K z^{2\nu}} \begin{pmatrix} \mathcal{O}(|z|^{2\nu}) + \mathcal{O}(|z|^\nu) & \mathcal{O}(|z|^{2\nu}) & \mathcal{O}(|z|^{2\nu}) + \mathcal{O}(|z|^\nu) \\ \mathcal{O}(|z|^{2\nu}) + \mathcal{O}(|z|^\nu) + \mathcal{O}(\log |z|) & \mathcal{O}(|z|^{2\nu}) & \mathcal{O}(|z|^{2\nu}) + \mathcal{O}(|z|^\nu) + \mathcal{O}(\log |z|) \\ \mathcal{O}(|z|^{2\nu}) + \mathcal{O}(|z|^\nu) & \mathcal{O}(|z|^{2\nu}) & \mathcal{O}(|z|^{2\nu}) + \mathcal{O}(|z|^\nu) \end{pmatrix},
\end{aligned}$$

where again the $\mathcal{O}(\log |z|)$ terms are absent if $\nu \neq 0$. Using this with (5.8) and (5.4), we arrive at (5.6) after a straightforward calculation. \square

5.2 Second transformation $X \mapsto U$

The second transformation $X \mapsto U$ serves to (partly) normalize the behavior at infinity. To do this we will use certain functions related to the vector equilibrium problem. We call these functions λ -functions and define them as a transformed version of the λ -functions in [23]. More precisely, denoting with $\lambda_j^{\text{NC}}(z)$, $j = 1, 2, 3, 4$, the λ -functions used in [23], we will work with the square root versions $\lambda_j(z) = 2\lambda_j^{\text{NC}}(\sqrt{z})$.

Lemma 5.3. *There exist functions λ_j , $j = 1, 2, 3, 4$, analytic on $\mathbb{C} \setminus \mathbb{R}$ that satisfy the following conditions.*

(a) *As $z \rightarrow \infty$ we have*

$$\begin{aligned}\lambda_1(z) &= V(z) - \log(z) - \ell_1 + \mathcal{O}(z^{-1}), \\ \lambda_2(z) &= \theta_1(z) + \frac{1}{3}\log(z) + Cz^{-1/3} + Dz^{-2/3} + \mathcal{O}(z^{-1}),\end{aligned}\tag{5.10}$$

$$\begin{aligned}\lambda_3(z) &= \theta_2(z) + \begin{cases} \frac{1}{3}\log(z) + C\omega z^{-1/3} + D\omega^2 z^{-2/3} + \mathcal{O}(z^{-1}) & \text{Im } z > 0, \\ \frac{1}{3}\log(z) + C\omega^2 z^{-1/3} + D\omega z^{-2/3} + \mathcal{O}(z^{-1}) & \text{Im } z < 0, \end{cases} \\ \lambda_4(z) &= \theta_3(z) + \begin{cases} \frac{1}{3}\log(z) + C\omega^2 z^{-1/3} + D\omega z^{-2/3} + \mathcal{O}(z^{-1}) & \text{Im } z > 0, \\ \frac{1}{3}\log(z) + C\omega z^{-1/3} + D\omega^2 z^{-2/3} + \mathcal{O}(z^{-1}) & \text{Im } z < 0, \end{cases}\end{aligned}\tag{5.11}$$

where C , D and ℓ_1 are real constants.

(b) *There exists a positive integer N , two sets of ordered numbers*

$$0 = b_0 \leq a_1 < b_1 < a_2 < \dots < a_N < b_N < a_{N+1} = \infty,$$

$$1 = \alpha_0 > \alpha_1 > \dots > \alpha_N = 0,$$

and constants $c_2 \geq 0$ and $c_3 \geq 0$ such that the λ -functions satisfy the following jump conditions:

(i) *On \mathbb{R}^+ we have*

$$\lambda_{1,\pm} - \lambda_{2,\mp} = 0 \quad \text{on } (a_j, b_j), \quad j = 1, \dots, N, \tag{5.12}$$

$$\lambda_{1,+} - \lambda_{1,-} = -2\pi i \alpha_j \quad \text{on } (b_j, a_{j+1}), \quad j = 0, \dots, N,$$

$$\lambda_{2,+} - \lambda_{2,-} = 2\pi i \alpha_j \quad \text{on } (b_j, a_{j+1}), \quad j = 0, \dots, N,$$

$$\lambda_{3,\pm} - \lambda_{4,\mp} = 0 \quad \text{on } (c_3, \infty), \tag{5.13}$$

$$\lambda_{3,+} - \lambda_{3,-} = -2\pi i/3 \quad \text{on } (0, c_3),$$

$$\lambda_{4,+} - \lambda_{4,-} = 2\pi i/3 \quad \text{on } (0, c_3).$$

(ii) *On \mathbb{R}^- we have*

$$\lambda_{1,+} - \lambda_{1,-} = -2\pi i \quad \text{on } \mathbb{R}^-, \tag{5.14}$$

$$\lambda_{2,\pm} - \lambda_{3,\mp} = \pm 2\pi i/3 \quad \text{on } (-\infty, -c_2),$$

$$\lambda_{2,+} - \lambda_{2,-} = 2\pi i/3 \quad \text{on } (-c_2, 0)$$

$$\lambda_{3,+} - \lambda_{3,-} = 2\pi i/3 \quad \text{on } (-c_2, 0)$$

$$\lambda_{4,+} - \lambda_{4,-} = 2\pi i/3 \quad \text{on } \mathbb{R}^-.$$

Proof. Define

$$\lambda_j(z) = 2\lambda_j^{\text{NC}}(\sqrt{z}), \quad j = 1, 2, 3, 4, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{5.15}$$

where we take the potentials as in (2.19). Then (a) follows from [23, Lemma 4.14] and (4.16). Note however that error terms stated there are not optimal. (b)(i) is direct from [23, Lemma 4.12]. To prove (ii) we need some preparations. By [23, Definition 4.1], it follows that for $x \geq 0$

$$\begin{aligned} g_1^{\text{NC}}(ix) - g_1^{\text{NC}}(-ix) &= \pi i, \\ g_{2,\pm}^{\text{NC}}(ix) - g_{2,\mp}^{\text{NC}}(-ix) &= 2\pi i/3, \\ g_3^{\text{NC}}(ix) - g_3^{\text{NC}}(-ix) &= \pi i/3, \end{aligned}$$

and

$$\begin{aligned} g_2^{\text{NC}}(-z) &= g_2^{\text{NC}}(z) \mp 2\pi i/3, & \pm \operatorname{Im} z > 0, \\ g_3^{\text{NC}}(-z) &= g_3^{\text{NC}}(z) \mp \pi i/3, & \pm \operatorname{Im} z > 0. \end{aligned}$$

Then [23, Definition 4.11] and (4.17) yield for $x \geq 0$

$$\begin{aligned} \lambda_1^{\text{NC}}(ix) - \lambda_1^{\text{NC}}(-ix) &= -\pi i, \\ \lambda_{2,-}^{\text{NC}}(ix) - \lambda_{2,-}^{\text{NC}}(-ix) &= \pm \pi i/3, & c_2^{\text{NC}} < x < +\infty, \\ \lambda_{2,-}^{\text{NC}}(ix) - \lambda_{2,-}^{\text{NC}}(-ix) &= \pi i/3, & 0 < x < c_2^{\text{NC}}, \\ \lambda_{3,-}^{\text{NC}}(ix) - \lambda_{3,-}^{\text{NC}}(-ix) &= \pi i/3, & 0 < x < c_2^{\text{NC}}, \\ \lambda_4^{\text{NC}}(ix) - \lambda_4^{\text{NC}}(-ix) &= \pi i/3. \end{aligned}$$

This, together with (5.15), implies (ii) in (b), where we take $c_2 = (c_2^{\text{NC}})^2$. \square

The λ -functions also satisfy a number of inequalities stated in the following lemma.

Lemma 5.4.

$$\operatorname{Re}(\lambda_{2,+} - \lambda_{1,-}) < 0, \quad \text{on } (b_j, a_{j+1}), \quad j = 0, \dots, N, \quad (5.16)$$

$$\operatorname{Re}(\lambda_{2,+} - \lambda_{3,-}) < 0, \quad \text{on } (-c_2, 0], \quad (5.17)$$

$$\operatorname{Re}(\lambda_{4,+} - \lambda_{3,-}) < 0, \quad \text{on } [0, c_3). \quad (5.18)$$

Proof. This follows from (5.15) and [23, Lemma 4.13]. \square

Remark 5.5. For the constants c_2 and c_3 in Lemma 5.3, we have $c_2 > -y^*(\alpha)$ and $c_3 = 0$ if $\alpha \geq 0$, and $c_3 < x^*(\alpha)$ if $\alpha < 0$, where $x^*(\alpha)$ and $y^*(\alpha)$ are given in Lemma 4.6. The functions λ_1 and λ_2 are defined and analytic on $\mathbb{C} \setminus (-\infty, b_N]$, whereas λ_3 and λ_4 are defined and analytic in $\mathbb{C} \setminus \mathbb{R}$.

For future reference we specify the behavior of the λ -functions near the origin.

Lemma 5.6. *In a neighborhood of the origin the λ -functions, defined in Lemma 5.3, have the following behavior for $z \rightarrow 0$.*

(a) *In Cases I and IV there exists a constant $c_1 > 0$ such that*

$$(\lambda_1 - \lambda_2)(z) = \mp 2\pi i \pm ic_1 z^{1/2} + \mathcal{O}(z), \quad \pm \operatorname{Im} z > 0.$$

(b) *In Case III there exists a constant $c_2 > 0$ such that*

$$(\lambda_2 - \lambda_3)(z) = \pm \frac{4\pi i}{3} + c_2 z^{1/2} + \mathcal{O}(z), \quad \pm \operatorname{Im} z > 0.$$

(c) *In Cases I and II there exists a constant $c_3 > 0$ such that*

$$(\lambda_3 - \lambda_4)(z) = \mp \frac{2\pi i}{3} \pm ic_3 z^{1/2} + \mathcal{O}(z), \quad \pm \operatorname{Im} z > 0.$$

Proof. (a), (b), and (c) follow from (5.15) and the proofs of [23, Lemma 7.5, Lemma 7.1, Lemma 7.2] respectively. \square

We can now define the transformation $X \mapsto U$ in a similar way as in [23].

Definition 5.7. We define the 4×4 matrix-valued function U by

$$U(z) = U_0 e^{nL} X(z) e^{nG(z)}, \quad (5.19)$$

where U_0 is a constant invertible matrix to be determined later,

$$G = \text{diag}(\lambda_1 - V, \lambda_2 - \theta_1, \lambda_3 - \theta_2, \lambda_4 - \theta_3),$$

and $L = \text{diag}(\ell_1, 0, 0, 0)$ with ℓ_1 the constant from Lemma 5.3.

RH problem 5.8. *The matrix-valued function U defined in (5.19) is the unique solution of the following RH problem.*

(1) $U(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.

(2) We have

$$U_+(x) = U_-(x) \begin{cases} \text{diag}((J_U)_1(x), (J_U)_3(x)), & x \in \mathbb{R}_+, \\ \text{diag}(e^{\nu\pi i}, (J_U)_2(x), e^{-\nu\pi i}), & x \in \mathbb{R}_-, \end{cases}$$

where

$$\begin{aligned} (J_U)_1 &= \begin{pmatrix} e^{n(\lambda_{1,+} - \lambda_{1,-})} & e^{n(\lambda_{2,+} - \lambda_{1,-})} \\ 0 & e^{n(\lambda_{2,+} - \lambda_{2,-})} \end{pmatrix}, \\ (J_U)_2 &= \begin{pmatrix} e^{-\nu\pi i} e^{n(\lambda_{2,+} - \lambda_{2,-})} & 0 \\ -e^{n(\lambda_{2,+} - \lambda_{3,-})} & e^{\nu\pi i} e^{n(\lambda_{3,+} - \lambda_{3,-})} \end{pmatrix}, \\ (J_U)_3 &= \begin{pmatrix} e^{n(\lambda_{3,+} - \lambda_{3,-})} & e^{n(\lambda_{4,+} - \lambda_{3,-})} \\ 0 & e^{n(\lambda_{4,+} - \lambda_{4,-})} \end{pmatrix}. \end{aligned}$$

(3) We can choose the constant matrix U_0 such that

$$\begin{aligned} U(z) &= [I + \mathcal{O}(z^{-1})] \text{diag}(1, z^{1/3}, z^{-1/3}, 1) \\ &\quad \times \text{diag}(z^{\nu/2}, z^{-\nu/6} A_{\pm}^{-T}) \text{diag}(1, 1, \sigma^{\pm}, \sigma^{\mp 1}), \end{aligned} \quad (5.20)$$

as $z \rightarrow \infty$ with $\pm \text{Im } z > 0$, where σ and A_{\pm} are defined in (4.41) and (4.43) respectively.

(4) $U(z)$ has the same behavior as $X(z)$ near the origin; see (5.6).

Proof. Noting that $e^{n(\lambda_{1,+} - \lambda_{1,-})} = e^{n(\lambda_{4,+} - \lambda_{4,-})} = 1$ on \mathbb{R}^- , due to Lemma 5.3(b) and our assumption that $n \equiv 0 \pmod{3}$, the jump condition for U in item (2) follows from a straightforward calculation.

To show the asymptotic behavior of U for large z in item (3), we see from Lemma 5.3(a) that

$$\begin{aligned} e^{nG(z)} &= \text{diag}(z^{-n} e^{-n\ell_1}, z^{n/3}, z^{n/3}, z^{n/3}) \left[I + nC \begin{pmatrix} 0 & 0 \\ 0 & \Omega_{\pm} \end{pmatrix} z^{-1/3} \right. \\ &\quad \left. + nE \begin{pmatrix} 0 & 0 \\ 0 & \Omega_{\mp} \end{pmatrix} z^{-2/3} + \text{diag}(\mathcal{O}(z^{-1}), \mathcal{O}(z^{-1}), \mathcal{O}(z^{-1}), \mathcal{O}(z^{-1})) \right], \end{aligned}$$

as $z \rightarrow \infty$ with $\pm \text{Im } z > 0$. Here $E = D + C^2/2$ and

$$\Omega_+ = \text{diag}(1, \omega, \omega^2), \quad \Omega_- = \text{diag}(1, \omega^2, \omega). \quad (5.21)$$

Then, by the asymptotic behavior of X in (5.5) and the definition of U in (5.19), we get

$$\begin{aligned} U(z) &= U_0 (I + \mathcal{O}(z^{-1})) \operatorname{diag} (1, z^{1/3}, z^{-1/3}, 1) \\ &\quad \times \operatorname{diag} (z^{\nu/2}, z^{-\nu/6} A_{\pm}^{-T}) \left[I + nC \begin{pmatrix} 0 & 0 \\ 0 & \Omega_{\pm} \end{pmatrix} z^{-1/3} + nE \begin{pmatrix} 0 & 0 \\ 0 & \Omega_{\mp} \end{pmatrix} z^{-2/3} \right. \\ &\quad \left. + \operatorname{diag} (\mathcal{O}(z^{-1}), \mathcal{O}(z^{-1}), \mathcal{O}(z^{-1}), \mathcal{O}(z^{-1})) \right] \times \operatorname{diag} (1, 1, \sigma^{\pm 1}, \sigma^{\mp 1}), \end{aligned}$$

as $z \rightarrow \infty$ with $\pm \operatorname{Im} z > 0$. We can move the terms within square brackets to the front at the expense of an extra constant contribution. This follows from the observation that

$$A_{\pm}^{-T} \Omega_{\pm} A_{\pm}^T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_{\pm}^{-T} \Omega_{\mp} A_{\pm}^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (5.22)$$

This gives

$$\begin{aligned} U(z) &= U_0 \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} + \mathcal{O}(z^{-1}) \right] \operatorname{diag} (1, z^{1/3}, z^{-1/3}, 1) \operatorname{diag} (z^{\nu/2}, z^{-\nu/6} A_{\pm}^{-T}) \\ &\quad \times \operatorname{diag} (1, 1, \sigma^{\pm 1}, \sigma^{\mp 1}), \quad (5.23) \end{aligned}$$

as $z \rightarrow \infty$ with $\pm \operatorname{Im} z > 0$. By setting U_0 to be the inverse of the constant matrix between the above square brackets, we arrive at the claim in item (3). \square

Recalling our assumption that $n \equiv 0 \pmod{3}$, we find from Lemma 5.3(b) that

$$(J_U)_1 = \begin{cases} \begin{pmatrix} e^{n(\lambda_1, + - \lambda_1, -)} & 1 \\ 0 & e^{n(\lambda_2, + - \lambda_2, -)} \end{pmatrix}, & \text{on } (a_j, b_j), j = 1, \dots, N, \\ \begin{pmatrix} e^{-2\pi i n \alpha_j} & e^{n(\lambda_2, + - \lambda_1, -)} \\ 0 & e^{2\pi i n \alpha_j} \end{pmatrix}, & \text{on } (b_j, a_{j+1}), j = 0, \dots, N. \end{cases} \quad (5.24)$$

$$(J_U)_2 = \begin{cases} \begin{pmatrix} e^{-\nu \pi i} e^{n(\lambda_2, + - \lambda_2, -)} & 0 \\ -1 & e^{\nu \pi i} e^{n(\lambda_3, + - \lambda_3, -)} \end{pmatrix}, & \text{on } (-\infty, -c_2), \\ \begin{pmatrix} e^{-\nu \pi i} & 0 \\ -e^{n(\lambda_2, + - \lambda_3, -)} & e^{\nu \pi i} \end{pmatrix}, & \text{on } (-c_2, 0), \end{cases} \quad (5.25)$$

$$(J_U)_3 = \begin{cases} \begin{pmatrix} 1 & e^{n(\lambda_4, + - \lambda_3, -)} \\ 0 & 1 \end{pmatrix}, & \text{on } (0, c_3), \\ \begin{pmatrix} e^{n(\lambda_3, + - \lambda_3, -)} & 1 \\ 0 & e^{n(\lambda_4, + - \lambda_4, -)} \end{pmatrix}, & \text{on } (c_3, \infty). \end{cases} \quad (5.26)$$

Note that the diagonal terms of $(J_U)_1$ are 1 on $(0, a_1)$ and (b_N, ∞) .

5.3 Third transformation $U \mapsto S$

In the third transformation we open lenses around $\bigcup_{j=1}^N (a_j, b_j)$, $(-\infty, -c_2)$ and $(c_3, +\infty)$, which are denoted by L_1 , L_2 , and L_3 , respectively. See Figure 7 for a plot of the lenses in Case IV if $N = 3$. These lenses are chosen such that the following estimates hold.

Lemma 5.9. *We can find a neighborhood L_1 of $\bigcup_{j=1}^N (a_j, b_j)$ such that*

$$\operatorname{Re}(\lambda_1 - \lambda_2)(z) < 0 \quad \text{for } z \in L_1 \setminus \mathbb{R}^+.$$

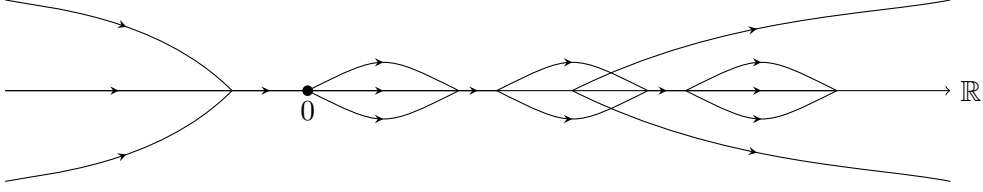


Figure 7: Jump contour Σ_S for S consisting of the lips of the lenses L_1, L_2, L_3 , and the axes. This is a picture of a Case IV situation with $N = 3$ since $a_1 = 0$ and $c_2, c_3 > 0$. The lips of the lens around $\text{supp}(\mu_3)$ intersect the lips of the lens around (a_k, b_k) if and only if $c_3 \in (a_k, b_k)$.

We can find a neighborhood L_2 of $(-\infty, -c_2)$ such that

$$\text{Re}(\lambda_3 - \lambda_2)(z) < 0 \quad \text{for } z \in L_2 \setminus \mathbb{R}^-,$$

and such that

$$\{z \in \mathbb{C} \mid \text{Re } z < -R, |\text{Im}(z)| < -\varepsilon \text{Re}(z)\} \subset L_2$$

for some $\varepsilon > 0$ and $R > 0$.

Finally, we can find a neighborhood L_3 of $(c_3, +\infty)$ such that

$$\text{Re}(\lambda_3 - \lambda_4)(z) < 0 \quad \text{for } z \in L_3 \setminus \mathbb{R}^+,$$

and such that

$$\{z \in \mathbb{C} \mid \text{Re } z > R, |\text{Im}(z)| < \varepsilon \text{Re}(z)\} \subset L_3$$

for some $\varepsilon > 0$ and $R > 0$.

We can assume that L_1, L_2, L_3 are symmetric with respect to complex conjugation. Moreover, we can assume that the intersection of the boundary ∂L_j with the upper (or lower) half plane is a collection of smooth curves, which can be oriented so that the real part strictly increases. We will often refer to ∂L_j as the lips of the lens L_j .

Proof. In view of the definition of λ -functions in (5.15), the statements about L_1, L_2 and L_3 are immediate from Lemmas 7.1, 7.2, and 7.5 in [23]. \square

To define the third transformation, we observe that in (5.24)–(5.26) we can factorize

$$\begin{aligned} (J_U)_1 &= \begin{pmatrix} 1 & 0 \\ e^{n(\lambda_1 - \lambda_2)_-} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{n(\lambda_1 - \lambda_2)_+} & 1 \end{pmatrix}, & \text{on } (a_j, b_j), \\ (J_U)_2 &= \begin{pmatrix} 1 & -e^{-\nu\pi i} e^{n(\lambda_3 - \lambda_2)_-} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -e^{\nu\pi i} e^{n(\lambda_3 - \lambda_2)_+} \\ 0 & 1 \end{pmatrix}, & \text{on } (-\infty, -c_2), \\ (J_U)_3 &= \begin{pmatrix} 1 & 0 \\ e^{n(\lambda_3 - \lambda_4)_-} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{n(\lambda_3 - \lambda_4)_+} & 1 \end{pmatrix}, & \text{on } (c_3, \infty), \end{aligned} \quad (5.27)$$

with the aid of (5.12), (5.13), (5.14) and the fact that n is a multiple of three.

We then define the third transformation $U \mapsto S$ by successively setting

$$T(z) = U(z) \times \begin{cases} I \pm e^{\pm\nu\pi i} e^{n(\lambda_3 - \lambda_2)(z)} E_{2,3}, & \text{for } z \in L_2 \cap \mathbb{C}^\pm, \\ I \mp e^{n(\lambda_3 - \lambda_4)(z)} E_{4,3}, & \text{for } z \in L_3 \cap \mathbb{C}^\pm, \\ I, & \text{elsewhere,} \end{cases} \quad (5.28)$$

and

$$S(z) = \begin{cases} T(z) \times (I \mp e^{n(\lambda_1 - \lambda_2)(z)} E_{2,1}), & \text{for } z \in L_1 \cap \mathbb{C}^\pm, \\ T(z), & \text{elsewhere,} \end{cases} \quad (5.29)$$

where \mathbb{C}^\pm denotes the upper or lower half plane respectively, and $E_{i,j}$ denotes the 4×4 elementary matrix of which all entries are 0, except for the (i, j) -th entry, which is 1. Then S is the unique solution of the following RH problem.

RH problem 5.10. *The matrix-valued function S satisfies the following conditions.*

- (1) $S(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma_S$, where Σ_S is the contour consisting of the real axis and the lips of the lenses L_i , $i = 1, 2, 3$, see Figure 7.
- (2) For $z \in \Sigma_S$, S has a jump

$$S_+(z) = S_-(z) \begin{cases} \text{diag}((J_S)_1(z), (J_S)_3(z)), & \text{for } z \text{ in } \mathbb{R}^+ \text{ and the lips of } L_1, L_3, \\ \text{diag}(e^{\nu\pi i}, (J_S)_2(z), e^{-\nu\pi i}), & \text{for } z \text{ in } \mathbb{R}^- \text{ and the lips of } L_2, \end{cases}$$

where

$$(J_S)_1 = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (a_j, b_j), j = 1, \dots, N, \\ \begin{pmatrix} e^{-2\pi i n \alpha_j} & e^{n(\lambda_{2,+} - \lambda_{1,-})} \\ 0 & e^{2\pi i n \alpha_j} \end{pmatrix}, & \text{on } (b_j, a_{j+1}), j = 0, \dots, N, \\ \begin{pmatrix} 1 & 0 \\ e^{n(\lambda_1 - \lambda_2)} & 1 \end{pmatrix}, & \text{on the lips of } L_1, \\ I_2, & \text{on the lips of } L_3, \end{cases}$$

$$(J_S)_2 = \begin{cases} \begin{pmatrix} e^{-\nu\pi i} & 0 \\ -e^{n(\lambda_{2,+} - \lambda_{3,-})} & e^{\nu\pi i} \end{pmatrix}, & \text{on } (-c_2, 0), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (-\infty, -c_2), \\ \begin{pmatrix} 1 & -e^{\pm\nu\pi i} e^{n(\lambda_3 - \lambda_2)} \\ 0 & 1 \end{pmatrix}, & \text{on the upper/lower lip of } L_2, \end{cases}$$

and

$$(J_S)_3 = \begin{cases} \begin{pmatrix} 1 & e^{n(\lambda_{4,+} - \lambda_{3,-})} \\ 0 & 1 \end{pmatrix}, & \text{on } (0, c_3), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (c_3, +\infty), \\ I_2, & \text{on the lips of } L_1, \\ \begin{pmatrix} 1 & 0 \\ e^{n(\lambda_3 - \lambda_4)} & 1 \end{pmatrix}, & \text{on the upper/lower lip of } L_3. \end{cases}$$

- (3) $S(z)$ has the same behavior for $z \rightarrow \infty$ as $U(z)$.
- (4) $S(z)$ has the same behavior near the origin as $U(z)$ (and $X(z)$), see (5.6), provided that $z \rightarrow 0$ outside the lenses that end in 0.

Proof. The jump condition is straightforward by (5.27) and item (2) in RH problem 5.8. The asymptotic behavior of S follows from the definition in (5.28)–(5.29) and the large z behavior of the λ -functions in (5.10)–(5.11). \square

In view of Lemmas 5.4 and 5.9 it is easily seen that each entry of J_S is either constant or exponentially small as $n \rightarrow +\infty$.

5.4 Global parametrix

In this section we look for a global parametrix $S^{(\infty)}$. This will be a good global approximation of the matrix-valued function S when n is large. Ignoring all exponentially small entries for $n \rightarrow \infty$ in J_S , we are led to the following model RH problem for $S^{(\infty)}$.

RH problem 5.11. *We look for a 4×4 matrix-valued function $S^{(\infty)}$ that satisfies the following conditions.*

(1) $S^{(\infty)}(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.

(2) For $x \in \mathbb{R}$, $S^{(\infty)}$ has a jump

$$S_+^{(\infty)}(x) = S_-^{(\infty)}(x) \begin{cases} \text{diag}((J_{S^{(\infty)}})_1(x), (J_{S^{(\infty)}})_3(x)), & \text{for } x \text{ in } \mathbb{R}^+, \\ \text{diag}(e^{\nu\pi i}, (J_{S^{(\infty)}})_2(x), e^{-\nu\pi i}), & \text{for } x \text{ in } \mathbb{R}^-, \end{cases}$$

where

$$(J_{S^{(\infty)}})_1 = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (a_j, b_j), j = 1, \dots, N, \\ \text{diag}(e^{-2\pi i \alpha_j}, e^{2\pi i \alpha_j}), & \text{on } (b_j, a_{j+1}), j = 1, \dots, N-1, \\ I_2, & \text{on } (0, a_1) \cup (b_N, +\infty), \end{cases}$$

$$(J_{S^{(\infty)}})_2 = \begin{cases} \text{diag}(e^{-\nu\pi i}, e^{\nu\pi i}), & \text{on } (-c_2, 0), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (-\infty, -c_2), \end{cases}$$

and

$$(J_{S^{(\infty)}})_3 = \begin{cases} I_2, & \text{on } (0, c_3), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (c_3, +\infty). \end{cases}$$

(3) As $z \rightarrow \infty$ with $\pm \text{Im } z > 0$, we have

$$S^{(\infty)}(z) = [I + \mathcal{O}(z^{-1})] \text{diag}(1, z^{1/3}, z^{-1/3}, 1) \text{diag}(z^{\nu/2}, z^{-\nu/6} A_{\pm}^{-T}) \times \text{diag}(1, 1, \sigma^{\pm 1}, \sigma^{\mp 1}).$$

In this section we will construct a solution to the above RH problem.

Transforming the global parametrix: $S^{(\infty)} \mapsto N_{\nu}$

We will look for a solution $S^{(\infty)}$ to RH problem 5.11 in the form

$$S^{(\infty)}(z) = N_{\nu}(z) \text{diag}\left(z^{\frac{\nu-1}{2}}, z^{\frac{1-\nu}{6}} I_3\right) \text{diag}(1, 1, \sigma^{\pm 1}, \sigma^{\mp 1}), \quad \pm \text{Im } z > 0. \quad (5.30)$$

Then N_{ν} must satisfy the following RH problem.

RH problem 5.12. We look for a 4×4 matrix-valued function N_{ν} satisfying the following conditions.

(1) $N_{\nu}(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.

(2) For $x \in \mathbb{R}$, N_{ν} has a jump

$$N_{\nu,+}(x) = N_{\nu,-}(x) \begin{cases} \text{diag}((J_{N_{\nu}})_1(x), (J_{N_{\nu}})_3(x)), & \text{for } x \text{ in } \mathbb{R}^+, \\ \text{diag}(-1, (J_{N_{\nu}})_2(x), -1), & \text{for } x \text{ in } \mathbb{R}^-, \end{cases}$$

where

$$(J_{N_{\nu}})_1 = \begin{cases} \begin{pmatrix} 0 & z^{\frac{2\nu-2}{3}} \\ -z^{\frac{2-2\nu}{3}} & 0 \end{pmatrix}, & \text{on } (a_j, b_j), j = 1, \dots, N, \\ \text{diag}(e^{-2\pi i \alpha_j}, e^{2\pi i \alpha_j}), & \text{on } (b_j, a_{j+1}), j = 1, \dots, N-1, \\ I_2, & \text{on } (0, a_1) \cup (b_N, +\infty), \end{cases}$$

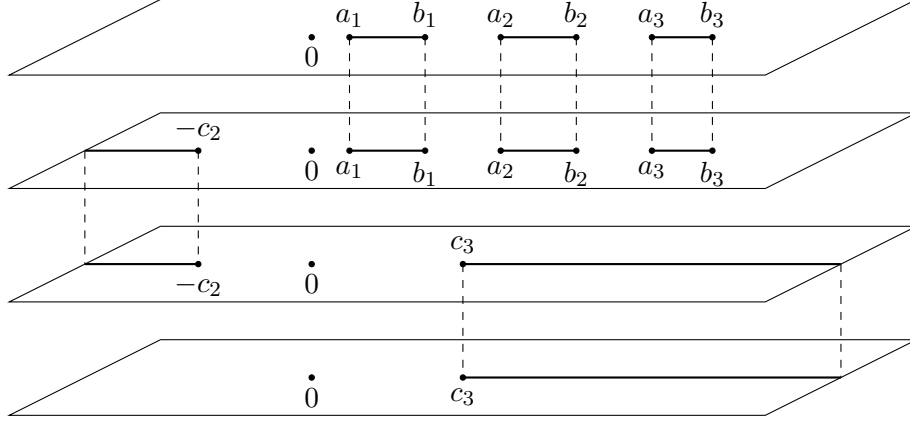


Figure 8: Plot of the Riemann surface \mathcal{R} for Case V (i.e. $a_1, c_2, c_3 > 0$) and genus $g = 2$.

$$(J_{N_\nu})_2 = \begin{cases} \text{diag} \left(e^{-\frac{\pi}{3}(1+2\nu)i}, e^{\frac{\pi}{3}(1+2\nu)i} \right), & \text{on } (-c_2, 0), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (-\infty, -c_2), \end{cases}$$

and

$$(J_{N_\nu})_3 = \begin{cases} \text{diag} \left(-e^{-\frac{\pi}{3}(1+2\nu)i}, -e^{\frac{\pi}{3}(1+2\nu)i} \right), & \text{on } (0, c_3), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (c_3, +\infty). \end{cases}$$

(3) As $z \rightarrow \infty$ with $\pm \text{Im } z > 0$, we have

$$N_\nu(z) = [I + \mathcal{O}(z^{-1})] \text{diag} \left(z^{1/2}, z^{1/6}, z^{-1/2}, z^{-1/6} \right) \text{diag} \left(1, A_\pm^{-T} \right). \quad (5.31)$$

Boundary value problem on a Riemann surface

The next step in the construction of the global parametrix is to find a ‘Szegő function’ on a certain Riemann surface. We define a four-sheeted Riemann surface \mathcal{R} as follows. We let \mathcal{R}_j , $j = 1, 2, 3, 4$, denote

$$\begin{aligned} \mathcal{R}_1 &= \mathbb{C} \setminus \bigcup_{k=1}^N [a_k, b_k], & \mathcal{R}_2 &= \mathbb{C} \setminus \left(\bigcup_{k=1}^N [a_k, b_k] \cup (-\infty, -c_2] \right), \\ \mathcal{R}_3 &= \mathbb{C} \setminus ((-\infty, -c_2] \cup [c_3, \infty)), & \mathcal{R}_4 &= \mathbb{C} \setminus [c_3, \infty). \end{aligned}$$

We connect the sheets \mathcal{R}_j , $j = 1, 2, 3, 4$, to each other in the usual crosswise manner, e.g. \mathcal{R}_1 is connected to \mathcal{R}_2 along the cuts $[a_k, b_k]$, $k = 1, \dots, N$. The Riemann surface is compactified by adding two points at infinity: ∞_1 is added to the first sheet while ∞_2 is common to the other sheets. We define \mathcal{B} as the union of four small disks, one around the origin of each sheet. We denote by $\mathcal{C} \subset \mathcal{R}$ the contour consisting of the intervals $[a_1, b_N]$ on the first two sheets, $[-c_2, 0]$ on sheets 2 and 3, and $[0, c_3]$ on sheets 3 and 4. See Figure 8 for an illustration in Case V.

We want to construct a Szegő function on this Riemann surface, i.e. we look for a scalar-valued function f on \mathcal{R} satisfying a boundary value problem. We denote the restriction of f to the j th sheet with $f_j(z)$, $j = 1, 2, 3, 4$.

Boundary value problem 5.13. *We look for a scalar-valued function f satisfying the following conditions.*

- (a) f is analytic on $\mathcal{R} \setminus \mathcal{C}$, and there exist constants C_1, C_2 such that $0 < C_1 < |f(z)| < C_2 < \infty$ on $\mathcal{R} \setminus (\mathcal{C} \cup \mathcal{B})$.

(b) f has the following jumps on \mathcal{C} :

$$f_{1,\pm}(x) = f_{2,\mp}(x)x^{\frac{2\nu-2}{3}}, \quad x \in (a_j, b_j), \quad (5.32)$$

$$f_{1,+}(x) = f_{1,-}(x)e^{-2\pi i\beta_j}, \quad x \in (b_j, a_{j+1}), \quad (5.33)$$

$$f_{2,+}(x) = f_{2,-}(x)e^{2\pi i\beta_j}, \quad x \in (b_j, a_{j+1}), \quad (5.34)$$

$$f_{2,+}(x) = f_{2,-}(x)e^{\frac{\pi i}{3}(1+2\nu)}, \quad x \in (-c_2, 0),$$

$$f_{3,+}(x) = f_{3,-}(x)e^{-\frac{\pi i}{3}(1+2\nu)}, \quad x \in (-c_2, 0),$$

$$f_{2,\pm}(x) = f_{3,\mp}(x), \quad x \in (-\infty, -c_2),$$

$$f_{3,+}(x) = f_{3,-}(x)e^{\frac{\pi i}{3}(1+2\nu)}, \quad x \in (0, c_3),$$

$$f_{4,+}(x) = f_{4,-}(x)e^{-\frac{\pi i}{3}(1+2\nu)}, \quad x \in (0, c_3),$$

$$f_{3,\pm}(x) = f_{4,\mp}(x), \quad x \in (c_3, \infty),$$

where in (5.33)–(5.34) we have certain real numbers β_j , $j = 1, \dots, N-1$, to be specified in Lemma 5.17.

(c) f is regular at $z = \infty$ in the sense that

$$f_1(z) = c + \mathcal{O}(z^{-1}), \quad c \neq 0,$$

and

$$\begin{pmatrix} f_2(z) \\ f_3(z) \\ f_4(z) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 z^{-1/3} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} + c_3 z^{-2/3} \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} + \mathcal{O}(z^{-1}), \quad c_1 \neq 0,$$

as $z \rightarrow \infty$ in the upper half plane.

(d) f has the following behavior around the origin of each sheet for $\text{Im } z > 0$. The behavior depends on the particular case we deal with, see Section 2.4.

$$f_1(z) = \begin{cases} \kappa_0 e^{\frac{i\pi}{4}(1-2\nu)} z^{\frac{1}{4}(2\nu-1)} (1 + \mathcal{O}(z^{1/2})), & \text{in Cases I and IV,} \\ \mathcal{O}(1), & \text{in Cases II, III, and V,} \end{cases} \quad (5.35)$$

$$f_2(z) = \begin{cases} \kappa_0 e^{-\frac{i\pi}{4}(1-2\nu)} z^{\frac{1}{12}(5-2\nu)} (1 + \mathcal{O}(z^{1/2})), & \text{in Cases I and IV,} \\ \kappa_1 z^{\frac{1}{6}(1+2\nu)} (1 + \mathcal{O}(z)), & \text{in Cases II and V,} \\ \kappa_2 z^{-\frac{1}{12}(1+2\nu)} (1 + \mathcal{O}(z^{1/2})), & \text{in Case III;} \end{cases} \quad (5.36)$$

$$f_3(z) = \begin{cases} \kappa_3 e^{-\frac{i\pi}{12}(1+2\nu)} z^{-\frac{1}{12}(1+2\nu)} (1 + \mathcal{O}(z^{1/2})), & \text{in Cases I and II,} \\ \kappa_2 e^{\frac{\pi i}{6}(1+2\nu)} z^{-\frac{1}{12}(1+2\nu)} (1 + \mathcal{O}(z^{1/2})), & \text{in Case III,} \\ \kappa_4 e^{\frac{\pi i}{6}(1+2\nu)} z^{-\frac{1}{3}(1+2\nu)} (1 + \mathcal{O}(z)), & \text{in Cases IV and V,} \end{cases} \quad (5.37)$$

$$f_4(z) = \begin{cases} \kappa_3 e^{\frac{i\pi}{12}(1+2\nu)} z^{-\frac{1}{12}(1+2\nu)} (1 + \mathcal{O}(z^{1/2})), & \text{in Cases I and II,} \\ \kappa_5 e^{-\frac{\pi i}{6}(1+2\nu)} z^{\frac{1}{6}(1+2\nu)} (1 + \mathcal{O}(z)), & \text{in Cases III, IV, and V,} \end{cases} \quad (5.38)$$

as $z \rightarrow 0$ with $\text{Im } z > 0$. The behavior in the lower half plane can be obtained using the symmetry condition $f(\bar{z}) = \overline{f(z)}$. The κ_j , $j = 0, \dots, 5$ are real constants.

Note that (c) is not an extra restriction as it is implied by (a).

We will settle the solvability of this boundary value problem at the end of this section.

Transforming the global parametrix: $N_\nu \mapsto M_\nu$

Assuming the solvability of the boundary value problem for the Szegő function f we can further reduce the RH problem for N_ν , i.e. we look for a solution to RH problem 5.12 in the form

$$N_\nu(z) = CM_\nu(z) \operatorname{diag} \left(\frac{1}{f_1(z)}, \frac{1}{f_2(z)}, \frac{1}{f_3(z)}, \frac{1}{f_4(z)} \right), \quad (5.39)$$

where C denotes the explicit constant matrix

$$C = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c_1 & c_3 & c_2 \\ 0 & 0 & c_1 & 0 \\ 0 & 0 & c_2 & c_1 \end{pmatrix}, \quad (5.40)$$

with c, c_1, c_2, c_3 the constants from condition (c) in Boundary Value Problem 5.13. The matrix C does not influence the jumps but will serve to get the appropriate asymptotics of $M_\nu(z)$ for large z . Putting $\tilde{\alpha}_j := \alpha_j + \beta_j/n$ for $j = 1, \dots, N-1$, the matrix-valued function M_ν must satisfy the following RH problem.

RH problem 5.14. *We look for a 4×4 matrix-valued function M_ν satisfying the following conditions.*

(1) $M_\nu(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.

(2) For $x \in \mathbb{R}$, M_ν has a jump

$$(M_\nu)_+(x) = (M_\nu)_-(x) \begin{cases} \operatorname{diag}((J_{M_\nu})_1(x), (J_{M_\nu})_3(x)), & \text{for } x \text{ in } \mathbb{R}^+, \\ \operatorname{diag}(-1, (J_{M_\nu})_2(x), -1), & \text{for } x \text{ in } \mathbb{R}^-, \end{cases} \quad (5.41)$$

where

$$(J_{M_\nu})_1 = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (a_j, b_j), j = 1, \dots, N, \\ \operatorname{diag}(e^{-2\pi i n \tilde{\alpha}_j}, e^{2\pi i n \tilde{\alpha}_j}), & \text{on } (b_j, a_{j+1}), j = 1, \dots, N-1, \\ I_2, & \text{on } (0, a_1) \cup (b_N, +\infty), \end{cases}$$

$$(J_{M_\nu})_2 = \begin{cases} I_2, & \text{on } (-c_2, 0), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (-\infty, -c_2), \end{cases}$$

and

$$(J_{M_\nu})_3 = \begin{cases} -I_2, & \text{on } (0, c_3), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (c_3, +\infty). \end{cases}$$

(3) As $z \rightarrow \infty$ and $\pm \operatorname{Im} z > 0$, we have

$$M_\nu(z) = [I + \mathcal{O}(z^{-1})] \operatorname{diag} \left(z^{1/2}, z^{1/6}, z^{-1/2}, z^{-1/6} \right) \operatorname{diag} \left(1, A_\pm^{-T} \right). \quad (5.42)$$

Indeed, the conditions (1) and (2) in the RH problem for M_ν are immediate from (5.39), RH problem 5.12, and Boundary Value Problem 5.13. For condition (3) one also uses the identities (5.21)–(5.22) and (5.40).

Constructing the global parametrix

We will immediately prove the solvability of RH problem 5.14 which then in combination with (5.30) and (5.39) finishes the construction of the global parametrix.

We solve RH problem 5.14 by reducing it to the RH problem for the global parametrix in the non-chiral two-matrix model described in [23, (8.1)–(8.6)]. The latter RH problem was solved in the same paper. We denote that solution here as $M^{\text{NC}}(z)$. Note that $M^{\text{NC}}(z)$ depends on certain parameters $n \in \mathbb{Z}$ and $0 < \alpha_1^{\text{NC}} < \dots < \alpha_{N-1}^{\text{NC}}$ that will be specified later. We will also need the symmetry relation

$$M^{\text{NC}}(-z) = \text{diag}(1, -1, 1, -1) M^{\text{NC}}(z) \text{diag}(1, -1, 1, -1), \quad (5.43)$$

which is not hard to verify. Now we claim that,

$$M_\nu(z) := KL \text{diag}(-z^{1/2}, 1, z^{-1/2}, 1) M^{\text{NC}}(z^{1/2}) \text{diag}(-1, -1, \mp 1, \pm 1), \quad \pm \text{Im } z > 0, \quad (5.44)$$

solves RH problem 5.14. Here K is a constant matrix of the form

$$K = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix}, \quad (5.45)$$

for certain constants $*$, serving to get the correct asymptotics for $z \rightarrow \infty$. Note that an apparent $z^{1/2}$ contribution for $z \rightarrow \infty$ vanishes due to (5.43). L is a matrix of the form

$$L = I + \kappa z^{-1} E_{3,1}, \quad (5.46)$$

for a suitable constant κ . The matrix L serves to get the correct behavior as $z \rightarrow 0$ and will be constructed in the proof of Lemma 5.15. A straightforward verification shows that M_ν defined in (5.44) indeed solves RH problem 5.14 for the right choice of parameters n and $0 < \alpha_1^{\text{NC}} < \dots < \alpha_{N-1}^{\text{NC}}$. Since the RH problem for M^{NC} is solvable for any choice of these parameters, see [23, Section 8], we have proved the solvability of the RH problem for M_ν .

Apart from the solvability of Boundary Value Problem 5.13 we have now finished the construction of the global parametrix by (5.30), (5.39), and (5.44). We will settle the solvability of Boundary Value Problem 5.13 at the very end of this section but first we will discuss the behavior of the global parametrix around the origin.

Behavior of $S^{(\infty)}$ near the origin

In the next lemma we discuss the behavior of $S^{(\infty)}$ near the origin.

Lemma 5.15. *The constant matrix L in (5.46) can be chosen such that $S^{(\infty)}$ has the following behavior near the origin $z = 0$:*

$$\begin{aligned} S^{(\infty)}(z) \text{diag}(z^{1/4}, z^{1/4}, z^{1/4}, z^{1/4}) &= \mathcal{O}(1), & \text{in Case I,} \\ S^{(\infty)}(z) \text{diag}(z^{-\nu/2}, z^{(1+\nu)/2}, z^{1/4}, z^{1/4}) &= \mathcal{O}(1), & \text{in Case II,} \\ S^{(\infty)}(z) \text{diag}(z^{-\nu/2}, z^{1/4}, z^{1/4}, z^{(1+\nu)/2}) &= \mathcal{O}(1), & \text{in Case III,} \\ S^{(\infty)}(z) \text{diag}(z^{1/4}, z^{1/4}, z^{-\nu/2}, z^{(1+\nu)/2}) &= \mathcal{O}(1), & \text{in Case IV,} \\ S^{(\infty)}(z) \text{diag}(z^{-\nu/2}, z^{(1+\nu)/2}, z^{-\nu/2}, z^{(1+\nu)/2}) &= \mathcal{O}(1), & \text{in Case V.} \end{aligned} \quad (5.47)$$

Proof. By combining the above transformations (5.30), (5.39) and (5.44) we get

$$\begin{aligned} S^{(\infty)}(z) &= CKL \text{diag}(-z^{1/2}, 1, z^{-1/2}, 1) M^{\text{NC}}(z^{1/2}) \text{diag}(-1, -1, \mp 1, \pm 1) \\ &\quad \times \text{diag}(f_1^{-1}(z), f_2^{-1}(z), \sigma^{\pm 1} f_3^{-1}(z), \sigma^{\mp 1} f_4^{-1}(z)) \text{diag}\left(z^{\frac{\nu-1}{2}}, z^{\frac{1-\nu}{6}} I_3\right), \end{aligned} \quad (5.48)$$

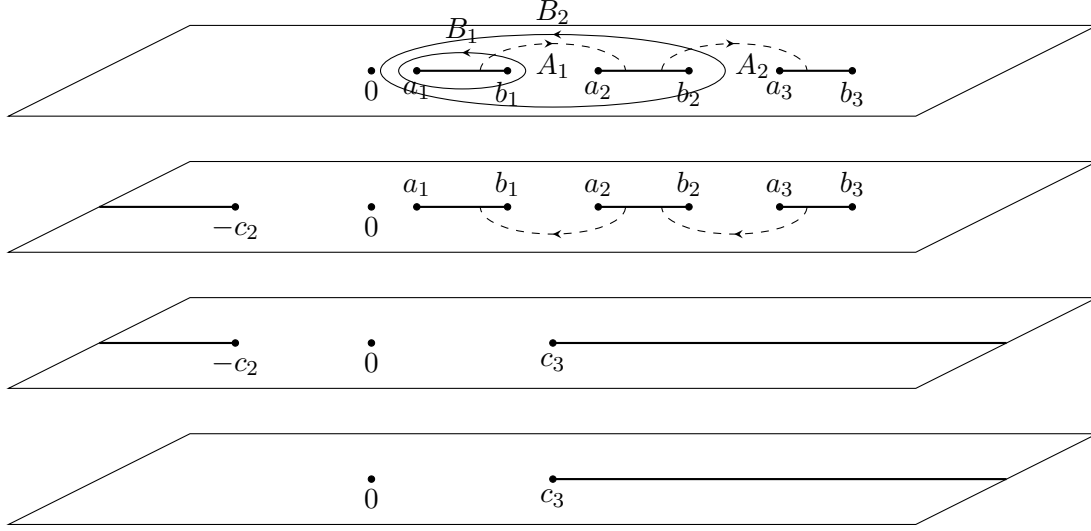


Figure 9: Plot of the Riemann surface \mathcal{R} and the canonical homology basis $(A_1, \dots, A_g, B_1, \dots, B_g)$ for Case V (i.e. $a_1, c_2, c_3 > 0$) and genus $g = 2$.

for $\pm \operatorname{Im} z > 0$.

From the construction in [23, Sec. 8] and the symmetry (5.43) it follows that

$$M^{\text{NC}}(z) = A + \mathcal{O}(z), \quad (5.49)$$

as $z \rightarrow 0$ in the first quadrant of \mathbb{C} , where

$$A = \begin{cases} \begin{pmatrix} a & a & * & * \\ b & -b & * & * \\ c & c & * & * \\ d & -d & * & * \end{pmatrix}, & \text{Cases I, IV,} \\ \begin{pmatrix} a & * & * & * \\ 0 & * & * & * \\ c & * & * & * \\ 0 & * & * & * \end{pmatrix}, & \text{Cases II, III, V,} \end{cases} \quad (5.50)$$

where a, b, c, d are certain constants with $abcd \neq 0$. We then define the matrix L as in (5.46) with the constant κ given by $\kappa = c/a$. The lemma then follows from a straightforward calculation using (5.48)–(5.50) and the behavior of the Szegő functions f_i near the origin in (5.35)–(5.38). \square

Remark 5.16. By using a finer analysis of the structure in (5.50), one can show that each of the terms $z^{(1+\nu)/2}$ in the Cases II–V in (5.47) can be replaced by $z^{\nu/2}$.

Solvability of Boundary Value Problem 5.13

It remains to prove the existence of the Szegő function f as a solution of Boundary Value Problem 5.13. On the Riemann surface \mathcal{R} we construct a canonical homology basis consisting of closed curves A_1, \dots, A_g and B_1, \dots, B_g with g the genus of \mathcal{R} . Here B_j is a closed curve on the first sheet going counterclockwise around the union of cuts $\bigcup_{k=1}^j [a_k, b_k]$, $j = 1, \dots, g$. On the other hand, A_j is a closed curve on the first two sheets whose intersection with the first sheet is a path connecting a point of the cut (a_j, b_j) to a point of the cut (a_{j+1}, b_{j+1}) , and whose intersection with the second sheet is the complex conjugate of this path, with the reverse orientation. See Figure 9 for an illustration in Case V.

Let $dw_1(z), \dots, dw_g(z)$ be a basis for the space of holomorphic differentials on \mathcal{R} normalized with respect to the cycles A_1, \dots, A_g , i.e., such that

$$\int_{A_j} dw_k(t) = \delta_{jk}, \quad j, k = 1, \dots, g,$$

where δ_{jk} denotes the Kronecker delta.

Let P be a point on the Riemann surface \mathcal{R} which lies in the upper half plane of one of the sheets \mathcal{R}_j , $j = 1, \dots, 4$. There exists a sufficiently small neighborhood U of P , lying entirely in the upper half plane of \mathcal{R}_j , such that the holomorphic differential dw_k allows the representation

$$dw_k(z) = \rho_k(z) dz, \quad z \in U, \quad (5.51)$$

for a certain analytic function ρ_k , where we use z as the complex coordinate on the sheet \mathcal{R}_j . Letting \bar{U} be the set of complex conjugate points of U lying on the same sheet \mathcal{R}_j , we then have from the symmetry under complex conjugation that

$$dw_k(z) = \overline{\rho_k(\bar{z})} dz, \quad z \in \bar{U}, \quad (5.52)$$

where we again use $z \in \bar{U}$ as the complex coordinate on the sheet \mathcal{R}_j .

Now we construct the Szegő function f satisfying (a)–(d) above following Zverovich [46]. In the language of [46, Page 135] we must find a solution to *Riemann's homogeneous problem* where $\Phi(p)$ is our function $f(z)$, L is our contour \mathcal{C} , and with the divisors \mathcal{D} and \mathcal{J} prescribing the singularities given by $\mathcal{D} = 1$ and (using additive notation)

$$\mathcal{J}^{-1} = \begin{cases} \min\{\frac{-1+2\nu}{4}, \frac{5-2\nu}{12}\}0_{1,2} + \frac{-1-2\nu}{12}0_{3,4}, & \text{Case I,} \\ \frac{1+2\nu}{6}0_2 + \frac{-1-2\nu}{12}0_{3,4}, & \text{Case II,} \\ \frac{-1-2\nu}{12}0_{2,3} + \frac{1+2\nu}{6}0_4, & \text{Case III,} \\ \min\{\frac{-1+2\nu}{4}, \frac{5-2\nu}{12}\}0_{1,2} + \frac{-1-2\nu}{3}0_3 + \frac{1+2\nu}{6}0_4, & \text{Case IV,} \\ \frac{1+2\nu}{6}0_2 + \frac{-1-2\nu}{3}0_3 + \frac{1+2\nu}{6}0_4, & \text{Case V.} \end{cases}$$

Thus the only singularities or poles of f are allowed at the points lying over the origin. The precise form of the divisor \mathcal{J} is due to (5.35)–(5.38).

Denote by $G(t)$ the multiplicative factors appearing in the jump conditions in part (b) of the above boundary value problem for f . Note that $G(t)$ satisfies a Hölder condition on each analytic arc of the contour \mathcal{C} , except possibly at the origin of the first two sheets, in the case where $a_1 = 0$. Thus the behavior at the origin needs to be analyzed separately.

Following [46, Page 137] we introduce the piecewise analytic function $X(q)$, $q \in \mathcal{R}$, as

$$X(q) := e^{\frac{1}{2\pi i} \int_{\mathcal{C}} \ln G(\tau) d\hat{\omega}_{qq_0}(\tau)}. \quad (5.53)$$

Here $d\hat{\omega}_{qq_0}(t)$ is the discontinuous analogue to the Cauchy kernel, $q_0 \notin \mathcal{C}$. We will show that $X(q)$ solves Boundary Value Problem 5.13 for well-chosen values of the constants β_j , $j = 1, \dots, g$.

Lemma 5.17. *There exist $\beta_j \in \mathbb{R}$, $j = 1, \dots, g$, such that the function $X(q)$ defined in (5.53) solves Boundary Value Problem 5.13.*

Proof. We first prove that $X(q)$ satisfies condition (b) of the boundary value problem. The function $X(q)$ is analytic on $\mathcal{R} \setminus (\mathcal{C} \cup \bigcup_{k=1}^g A_k)$ and has the following jumps, see [46]

$$X_+(t) = X_-(t)G(t), \quad \text{for } t \in \mathcal{C}, \quad (5.54)$$

$$X_+(t) = X_-(t)e^{-\int_{\mathcal{C}} \ln G(\tau) dw_k(\tau)}, \quad \text{for } t \in A_k, \quad k = 1, \dots, g. \quad (5.55)$$

Note that the jump on A_k , $k = 1, \dots, g$, is constant. We claim that we can define the real constants β_j such that these jumps are actually trivial. To see this we first show that

$$\int_{\mathcal{C}} \ln G(\tau) dw_k(\tau), \quad k = 1, \dots, g, \quad (5.56)$$

is real. This follows from the symmetry under complex conjugation and the particular form of the jump factors $G(z)$. Indeed, using (5.32) and (5.51)–(5.52) the contribution of the cut $[a_j, b_j]$ on (5.56) can be written as

$$\begin{aligned} \frac{1}{2\pi i} \int_{[a_j, b_j]} \frac{2\nu - 2}{3} \log(x+) dw_k(x+) - \frac{1}{2\pi i} \int_{[a_j, b_j]} \frac{2\nu - 2}{3} \log(x-) dw_k(x-) \\ = \frac{1}{2\pi i} \frac{2\nu - 2}{3} \int_{a_j}^{b_j} \log(x) (\rho_k(x) - \overline{\rho_k(x)}) dx \in \mathbb{R}, \end{aligned}$$

where the first (second) integral is over the interval $[a_j, b_j]$ in the upper (lower) half plane of the first sheet. Also all contributions

$$\frac{1}{2\pi i} \int_K \log G(x) dw_k(x),$$

are real, where K is any of the intervals (b_j, a_j) , $j = 1, \dots, N - 1$, on the first or second sheet, $(-c_2, 0)$ on the second or third sheet, or $(0, c_3)$ on the third or fourth sheet. This follows from the fact that on these intervals the meromorphic differential dw_k is real and the particular form of $\log G(x)$ on these intervals. Moreover, in (5.56), the contribution from the gaps (b_j, a_{j+1}) on the first two sheets is given by (use (5.33)–(5.34))

$$\frac{1}{2\pi i} \int_{A_k} (-2\pi i \beta_j) dw_k(t) = -\beta_j \delta_{k,j}.$$

Hence, the constants β_j , $j = 1, \dots, g$, in (5.33)–(5.34) can indeed be chosen so that (5.56) is zero. Therefore the function $X(q)$ already satisfies condition (b) of Boundary Value Problem 5.13.

Next we prove condition (d). It follows from the precise definition of $G(z)$ that the quantities χ_k , χ in [46, Page 138] are all zero except at the points lying over the origin. Hence the function $X(q)$ can only have singularities at these points. The order of these singularities is described by the divisor \mathcal{E} in [46, Page 138]. In our case this divisor is precisely the inverse of \mathcal{J} above: $\mathcal{E} = \mathcal{J}^{-1}$. Indeed, this follows from the formulas in [46] except for the point $0_{1,2}$ in Cases I and IV, since there the jump matrices do not satisfy the boundedness and/or Hölder conditions. In Cases I and IV we have that $a_1 = 0$ and $c_2 > 0$, so in the union of disks \mathcal{B} the Szegő function f has the jumps

$$f_{1,\pm}(x) = f_{2,\mp}(x) x^{\frac{2\nu-2}{3}}, \quad x > 0, \quad (5.57)$$

$$f_{2,+}(x) = f_{2,-}(x) e^{\frac{\pi i}{3}(1+2\nu)}, \quad x < 0. \quad (5.58)$$

We define the conformal map

$$\zeta = \begin{cases} z^{1/2} & z \in \mathcal{R}_1 \cap \mathcal{B}, \\ -z^{1/2} & z \in \mathcal{R}_2 \cap \mathcal{B}, \end{cases}$$

where the cut of the fractional power is chosen along the positive real line, i.e. $0 < \arg z < 2\pi$. This map sends $(\mathcal{R}_1 \cup \mathcal{R}_2) \cap \mathcal{B}$ to a disk \mathcal{B}^* centered at the origin of the complex ζ -plane. Lifting the Szegő function to the ζ -plane we get

$$f(\zeta) = \begin{cases} f_1(\zeta^2) & \operatorname{Im} \zeta > 0, \\ f_2(\zeta^2) & \operatorname{Im} \zeta < 0. \end{cases}$$

Then (5.57)–(5.58) translate into

$$\begin{aligned} f_+(\zeta) &= f_-(\zeta)|\zeta|^{\frac{4\nu-4}{3}}, & \zeta \in \mathbb{R} \cap \mathcal{B}^*, \\ f_+(\zeta) &= f_-(\zeta)e^{\frac{\pi i}{3}(1+2\nu)}, & \zeta \in i\mathbb{R}^- \cap \mathcal{B}^*, \end{aligned}$$

where the part of the real line is oriented from left to right and the part of the imaginary axis from bottom to top. Applying log at both sides of the equation leads to an additive problem that can be solved using Cauchy transforms and the Sokhotski-Plemelj formula

$$\log f(\zeta) = \frac{2\nu-2}{3\pi i} \int_{-1}^1 \frac{\log |t|}{t-\zeta} dt + \frac{1+2\nu}{6} \int_{-i}^0 \frac{1}{t-\zeta} dt + h(\zeta), \quad \zeta \in \mathcal{B}^* \setminus (\mathbb{R} \cup i\mathbb{R}^-), \quad (5.59)$$

where h is meromorphic in \mathcal{B}^* with no poles except possibly at zero. The asymptotic behavior of the integrals in (5.59) as $\zeta \rightarrow 0$ can be calculated, e.g. using Mathematica, and is given by

$$\begin{aligned} \int_{-1}^1 \frac{\log |t|}{t-\zeta} dt &= \pm \pi i \log \zeta + \frac{\pi^2}{2} + \mathcal{O}(\zeta), & \text{as } \zeta \rightarrow 0, \zeta \in \mathbb{C}^\pm \\ \int_{-i}^0 \frac{1}{t-\zeta} dt &= \begin{cases} \log \zeta - \frac{\pi i}{2} + \mathcal{O}(\zeta), & \text{as } \zeta \rightarrow 0, \zeta \in I \cup II \cup IV, \\ \log \zeta + \frac{3\pi i}{2} + \mathcal{O}(\zeta) & \text{as } \zeta \rightarrow 0, \zeta \in III. \end{cases} \end{aligned}$$

Here I, II, III, IV denote the open quadrants of the complex ζ -plane. Plugging in these asymptotics in (5.59) we get

$$\log f(\zeta) = \begin{cases} \frac{1}{2}(2\nu-1) \log \zeta + \frac{i\pi}{4}(1-2\nu) + h(\zeta) + \mathcal{O}(\zeta), & \text{as } \zeta \rightarrow 0, \zeta \in I \cup II, \\ \frac{1}{6}(5-2\nu) \log \zeta + \frac{i\pi}{12}(7+2\nu) + h(\zeta) + \mathcal{O}(\zeta), & \text{as } \zeta \rightarrow 0, \zeta \in III, \\ \frac{1}{6}(5-2\nu) \log \zeta + \frac{i\pi}{4}(1-2\nu) + h(\zeta) + \mathcal{O}(\zeta), & \text{as } \zeta \rightarrow 0, \zeta \in IV. \end{cases}$$

In terms of the original functions f_1, f_2 (assuming that h is analytic) this precisely turns into (5.35)–(5.36) for Cases I and IV. This concludes the proof of the asymptotic behavior of the Szegő function around the origin.

Summarizing, we have now proved that $f(q) := X(q)$ solves Boundary Value Problem 5.13, where in (a) we only established the upper bound $|f(q)| < C_2 < +\infty$ on $\mathcal{R} \setminus \mathcal{B}$. To obtain the lower bound $0 < C_1 < |f(q)|$ on $\mathcal{R} \setminus \mathcal{B}$, let us first define the function $\tilde{f}(q)$ on \mathcal{R} which is the solution to the same boundary value problem as $f(q)$ except that each of the multiplicative jumps $G(t)$ is replaced by its inverse $1/G(t)$ and the asymptotic behavior at 0 is inverted as well. Proceeding as above we find such a solution $\tilde{f}(q)$ which is bounded, i.e., $|\tilde{f}(q)| < \tilde{C}_2 < +\infty$ on $\mathcal{R} \setminus \mathcal{B}$. But then the product function $f(q)\tilde{f}(q)$ is analytic and bounded on \mathcal{R} so by Liouville's theorem it must be a constant. Clearly it cannot be identically zero so it is a nonzero constant $C \neq 0$. This shows that $\tilde{f}(q) = C/f(q)$ and from the boundedness of \tilde{f} we then obtain the desired lower bound $0 < C_1 < |f(q)|$ with $C_1 = C/\tilde{C}_2$. \square

5.5 Local parametrices near the nonzero branch points

Near each of the branch points in $\{a_j, b_j \mid j = 1, \dots, N\} \cup \{-c_2, c_3\} \setminus \{0\}$, a local parametrix S^{Airy} can be built in the standard way with the help of Airy functions, see e.g. [23]. We omit the details here.

5.6 Local parametrix near the origin

In this section we construct the local parametrix $S^{(0)}$ near the origin. We will show that the local RH problem can be reduced to the RH problem in [17, Section 5.6]. As a first step we perform a preliminary transformation on RH problem 5.10.

We define the 4×4 matrix-valued function P by

$$P(z) = S(-z) \operatorname{diag} \left(e^{\pm \nu \pi i / 2}, e^{\mp \nu \pi i / 2}, e^{\pm \nu \pi i / 2}, e^{\mp \nu \pi i / 2} \right) J, \quad \text{for } \pm \operatorname{Im} z > 0, \quad (5.60)$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (5.61)$$

Let us denote by $\tilde{L}_i = -L_i$, $i = 1, 2, 3$, the lens around $\bigcup_{j=1}^N (-b_j, -a_j)$, $(c_2, +\infty)$ and $(-\infty, -c_3)$, respectively. We also introduce

$$\tilde{\lambda}_i(z) = \lambda_{5-i}(-z), \quad i = 1, 2, 3, 4. \quad (5.62)$$

Then P solves the following RH problem.

RH problem 5.18. *The matrix-valued function P satisfies the following conditions.*

- (1) $P(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma_P$, where Σ_P is the contour consisting of the real axis and the lips of the lenses \tilde{L}_i , $i = 1, 2, 3$.
- (2) For $z \in \Sigma_P$, P has a jump

$$P_+(z) = P_-(z) \begin{cases} \operatorname{diag} (1, (J_P)_2, 1), & \text{for } z \text{ in } \mathbb{R}^+ \text{ and the lips of } \tilde{L}_2, \\ \operatorname{diag} ((J_P)_1, (J_P)_3), & \text{for } z \text{ in } \mathbb{R}^- \text{ and the lips of } \tilde{L}_1, \tilde{L}_3, \end{cases}$$

where

$$(J_P)_1 = \begin{cases} \begin{pmatrix} e^{-\nu \pi i} & 0 \\ -e^{n(\tilde{\lambda}_1, -\tilde{\lambda}_{2,+})} & e^{\nu \pi i} \end{pmatrix}, & \text{on } (-c_3, 0), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (-\infty, -c_3), \\ I_2, & \text{on the lips of } \tilde{L}_1, \\ \begin{pmatrix} 1 & -e^{\pm \nu \pi i} e^{n(\tilde{\lambda}_2 - \tilde{\lambda}_1)} \\ 0 & 1 \end{pmatrix}, & \text{on the upper/lower lip of } \tilde{L}_3, \end{cases}$$

$$(J_P)_2 = \begin{cases} \begin{pmatrix} 1 & e^{n(\tilde{\lambda}_3, -\tilde{\lambda}_{2,+})} \\ 0 & 1 \end{pmatrix}, & \text{on } (0, c_2), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (c_2, \infty), \\ \begin{pmatrix} 1 & 0 \\ e^{n(\tilde{\lambda}_2 - \tilde{\lambda}_3)} & 1 \end{pmatrix}, & \text{on the lips of } \tilde{L}_2, \end{cases}$$

and

$$(J_P)_3 = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (-b_j, -a_j), j = 1, \dots, N, \\ \begin{pmatrix} e^{-\nu \pi i - 2\pi i n \alpha_j} & 0 \\ -e^{n(\tilde{\lambda}_3, -\tilde{\lambda}_{4,+})} & e^{\nu \pi i + 2\pi i n \alpha_j} \end{pmatrix}, & \text{on } (-a_{j+1}, -b_j), j = 0, \dots, N, \\ \begin{pmatrix} 1 & -e^{\pm \nu \pi i} e^{n(\tilde{\lambda}_4 - \tilde{\lambda}_3)} \\ 0 & 1 \end{pmatrix}, & \text{on the upper/lower lip of } \tilde{L}_1, \\ I_2, & \text{on the lips of } \tilde{L}_3. \end{cases}$$

(3) If $z \rightarrow 0$ outside the lenses that end in 0, we have

$$\begin{cases} P(z) \operatorname{diag}(|z|^{\nu/2}, |z|^{-\nu/2}, |z|^{\nu/2}, |z|^{-\nu/2}) = \mathcal{O}(1), & \text{if } \nu > 0, \\ P(z) \operatorname{diag}((\log |z|)^{-1}, 1, (\log |z|)^{-1}, 1) = \mathcal{O}(1), & \text{if } \nu = 0, \\ P(z) = \mathcal{O}(|z|^{\nu/2}), \quad P^{-1}(z) = \mathcal{O}(|z|^{\nu/2}), & \text{if } -1 < \nu < 0. \end{cases} \quad (5.63)$$

We can apply the transformation (5.60) to the global parametrix $S^{(\infty)}$ as well. That is, we define $S^{(\infty)} \mapsto P^{(\infty)}$ by

$$P^{(\infty)}(z) = S^{(\infty)}(-z) \operatorname{diag}\left(e^{\pm\nu\pi i/2}, e^{\mp\nu\pi i/2}, e^{\pm\nu\pi i/2}, e^{\mp\nu\pi i/2}\right) J, \quad \text{for } \pm \operatorname{Im} z > 0, \quad (5.64)$$

with J as in (5.60). Then the jumps for $P^{(\infty)}$ equal the jumps for P above, but with all the exponentially decaying entries of the form $e^{n(\tilde{\lambda}_j - \tilde{\lambda}_k)}$ removed. The behavior of $P^{(\infty)}$ near the origin follows trivially from (5.47).

Now we observe that the jumps in the RH problems for P and $P^{(\infty)}$ are reminiscent of those in [17, Section 5.6], with the variable α in the latter paper playing the role of our ν . Our construction of the local parametrix $P^{(0)}$ will be inspired by [17].

In the construction we have to make a case distinction between $\nu < 0$ and $\nu \geq 0$. To that end we define $\mathbf{1}_{\nu < 0} = 1$ if $\nu < 0$ and 0 if $\nu \geq 0$. The local parametrix $P^{(0)}$ is defined in a fixed but sufficiently small disk $D(0, \delta)$ around the origin, with radius $\delta > 0$. It satisfies the following RH problem.

RH problem 5.19. *We look for a 4×4 matrix-valued function $P^{(0)}(z) : D(0, \delta) \setminus \Sigma_P \rightarrow \mathbb{C}^{4 \times 4}$ satisfying the following conditions.*

- (1) $P^{(0)}(z)$ is analytic for $z \in D(0, \delta) \setminus \Sigma_P$.
- (2) For $z \in D(0, \delta) \cap \Sigma_P$, $P^{(0)}$ has a jump

$$P_+^{(0)}(z) = P_-^{(0)}(z) \begin{cases} \operatorname{diag}((J_{P^{(0)}})_1, (J_{P^{(0)}})_3), & \text{for } z \text{ in } (-\delta, 0) \text{ and the lips} \\ & \text{of } \tilde{L}_1, \tilde{L}_3, \\ \operatorname{diag}(1, (J_{P^{(0)}})_2, 1), & \text{for } z \text{ in } (0, \delta) \text{ and the lips of} \\ & \tilde{L}_2, \end{cases}$$

where

$$(J_{P^{(0)}})_1 = \begin{cases} \begin{pmatrix} e^{-\nu\pi i} & 0 \\ -\mathbf{1}_{\nu < 0} e^{n(\tilde{\lambda}_1, -\tilde{\lambda}_{2,+})} & e^{\nu\pi i} \end{pmatrix}, & \text{on } (-\delta, 0) \text{ in Cases III, IV, V,} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (-\delta, 0) \text{ in Cases I, II,} \\ I_2, & \text{on } D(0, \delta) \cap \text{the lips of } \tilde{L}_1, \\ \begin{pmatrix} 1 & -e^{\pm\nu\pi i} e^{n(\tilde{\lambda}_2 - \tilde{\lambda}_1)} \\ 0 & 1 \end{pmatrix}, & \text{on } D(0, \delta) \cap \text{the upper/lower lip of } \tilde{L}_3, \end{cases}$$

$$(J_{P^{(0)}})_2 = \begin{cases} \begin{pmatrix} 1 & \mathbf{1}_{\nu < 0} e^{n(\tilde{\lambda}_3, -\tilde{\lambda}_{2,+})} \\ 0 & 1 \end{pmatrix}, & \text{on } (0, \delta) \text{ in Cases I, II, IV, V,} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (0, \delta) \text{ in Case III,} \\ \begin{pmatrix} 1 & 0 \\ e^{n(\tilde{\lambda}_2 - \tilde{\lambda}_3)} & 1 \end{pmatrix}, & \text{on } D(0, \delta) \cap \text{the lips of } \tilde{L}_2, \end{cases}$$

and

$$(J_{P^{(0)}})_3 = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (-\delta, 0) \text{ in Cases I, IV,} \\ \begin{pmatrix} e^{-\nu\pi i} & 0 \\ -\mathbf{1}_{\nu < 0} e^{n(\tilde{\lambda}_3, -\tilde{\lambda}_4, +)} & e^{\nu\pi i} \end{pmatrix}, & \text{on } (-\delta, 0) \text{ in Cases II, III, V,} \\ \begin{pmatrix} 1 & -e^{\pm\nu\pi i} e^{n(\tilde{\lambda}_4 - \tilde{\lambda}_3)} \\ 0 & 1 \end{pmatrix}, & \text{on } D(0, \delta) \cap \text{the upper/lower lip of } \tilde{L}_1, \\ I_2, & \text{on } D(0, \delta) \cap \text{the lips of } \tilde{L}_3. \end{cases}$$

(3) If $z \rightarrow 0$ outside the lenses that end in 0, we have

$$\begin{cases} P^{(0)}(z) \operatorname{diag}(|z|^{\nu/2}, |z|^{-\nu/2}, |z|^{\nu/2}, |z|^{-\nu/2}) = \mathcal{O}(1), & \text{if } \nu > 0, \\ P^{(0)}(z) \operatorname{diag}((\log |z|)^{-1}, 1, (\log |z|)^{-1}, 1) = \mathcal{O}(1), & \text{if } \nu = 0, \\ P^{(0)}(z) = \mathcal{O}(|z|^{\nu/2}), \quad (P^{(0)})^{-1}(z) = \mathcal{O}(|z|^{\nu/2}), & \text{if } -1 < \nu < 0. \end{cases} \quad (5.65)$$

(4) On the boundary of $D(0, \delta)$ we have the uniform estimate

$$P^{(0)}(z) = P^{(\infty)}(z)(I + \mathcal{O}(1/n)), \quad n \rightarrow \infty. \quad (5.66)$$

Note that the exponentially small entries in the jump matrices on $(-\delta, 0) \cup (0, \delta)$ in the above RH problem are only present if $\nu < 0$; we neglect them if $\nu \geq 0$. The reason for this case distinction between $\nu < 0$ and $\nu \geq 0$ is explained in [17] (with there α playing the role of our ν); see also the estimates in Section 5.7 below.

To construct $P^{(0)}$, we need the model RH problem for the modified Bessel function.

RH problem 5.20. Denoting with γ_j , $j = 1, 2, 3$ the complex rays $\{\zeta \in \mathbb{C} \mid \arg \zeta = (j+1)\pi/3\}$, we look for a 2×2 matrix-valued function Ψ^{Bessel} such that

(1) Ψ^{Bessel} is analytic in $\mathbb{C} \setminus \bigcup_{j=1}^3 \gamma_j$.

(2) With the rays γ_j , $j = 1, 2, 3$ all oriented towards the origin, Ψ^{Bessel} has the jumps

$$\Psi_+^{\text{Bessel}} = \Psi_-^{\text{Bessel}} \times \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{\nu\pi i} & 1 \end{pmatrix}, & \text{on } \gamma_1, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } \gamma_2, \\ \begin{pmatrix} 1 & 0 \\ e^{-\nu\pi i} & 1 \end{pmatrix}, & \text{on } \gamma_3. \end{cases}$$

(3) Uniformly for $\zeta \rightarrow \infty$ we have

$$\Psi^{\text{Bessel}}(\zeta) = (2\pi\zeta^{1/2})^{-\sigma_3/2} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} + \mathcal{O}(\zeta^{-1/2}) \right) e^{2\zeta^{1/2}\sigma_3}, \quad (5.67)$$

with $\sigma_3 = \operatorname{diag}(1, -1)$.

(4) As $\zeta \rightarrow 0$ in $|\arg \zeta| < 2\pi/3$ we have

$$\Psi^{\text{Bessel}}(\zeta) = \begin{cases} \mathcal{O} \begin{pmatrix} \zeta^{\nu/2} & \zeta^{-\nu/2} \\ \zeta^{\nu/2} & \zeta^{-\nu/2} \end{pmatrix}, & \text{if } \nu > 0, \\ \mathcal{O} \begin{pmatrix} 1 & \log |\zeta| \\ 1 & \log |\zeta| \end{pmatrix}, & \text{if } \nu = 0, \\ \mathcal{O}(\zeta^{\nu/2}), & \text{if } \nu < 0. \end{cases} \quad (5.68)$$

This RH problem has a unique solution which is given in terms of modified Bessel and Hankel functions, see [34].

We are now ready to construct $P^{(0)}$ case by case, following the lines in [17]. First we give the construction for **Case I**. We may assume without loss of generality that the lips of \tilde{L}_1 and \tilde{L}_3 coincide within $D(0, \delta)$. If $\nu \geq 0$, we consider the functions

$$\phi_1(z) := \left(\tilde{\lambda}_1(z) - \tilde{\lambda}_2(z) \pm \frac{2\pi i}{3} \right)^2, \quad \phi_3(z) := \left(\tilde{\lambda}_3(z) - \tilde{\lambda}_4(z) \pm 2\pi i \right)^2, \quad \pm \operatorname{Im} z > 0. \quad (5.69)$$

It follows from (5.62) and Lemma 5.6 that these functions have analytic continuations to $D(0, \delta)$ that give conformal maps from a neighborhood of the origin onto itself, such that $\phi_i(x)$, $i = 1, 3$, is real and positive for $x \in (0, \delta)$. We deform the lips of \tilde{L}_1 (\tilde{L}_3) near 0 such that ϕ_i maps the upper and lower lips of \tilde{L}_1 (\tilde{L}_3) to the rays with angles $2\pi/3$ and $-2\pi/3$, respectively.

We now define

$$\begin{aligned} \hat{P}^{(0)}(z) = E(z) \operatorname{diag} \left(\sigma_1 \Psi^{\text{Bessel}} \left(\frac{n^2 \phi_1(z)}{16} \right) \sigma_1, \sigma_1 \Psi^{\text{Bessel}} \left(\frac{n^2 \phi_3(z)}{16} \right) \sigma_1 \right) \\ \times \operatorname{diag}(\sigma_3 e^{\frac{n}{2}(\tilde{\lambda}_1(z) - \tilde{\lambda}_2(z))\sigma_3}, \sigma_3 e^{\frac{n}{2}(\tilde{\lambda}_3(z) - \tilde{\lambda}_4(z))\sigma_3}), \end{aligned} \quad (5.70)$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_3 = \operatorname{diag}(1, -1)$, and where the prefactor $E(z)$ is analytic in $D(0, \delta)$ and is chosen to satisfy the matching condition on $\partial D(0, \delta)$, see below. We use the hat superscript to emphasize that we are in the situation $\nu \geq 0$. Note that $\hat{P}^{(0)}$ essentially decouples into two blocks containing the model RH problem for the modified Bessel function. With this definition, and assuming $n \equiv 0 \pmod{3}$, the items (1), (2), and (3) in the RH problem for $P^{(0)}$ are satisfied, by virtue of items (1), (2), and (4) in the RH problem for Ψ^{Bessel} .

To achieve the matching condition in item (4) of the RH problem for $P^{(0)}(z)$, we take $E(z)$ in (5.70) as

$$\begin{aligned} E(z) = P^{(\infty)}(z)(-1)^n \operatorname{diag}(\sigma_3, \sigma_3) \operatorname{diag} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \right) \\ \times \operatorname{diag} \left(\left(\frac{\pi n}{2} \phi_1^{1/2}(z) \right)^{-\sigma_3/2}, \left(\frac{\pi n}{2} \phi_3^{1/2}(z) \right)^{-\sigma_3/2} \right). \end{aligned} \quad (5.71)$$

Obviously $E(z)$ is analytic for $z \in D(0, \delta) \setminus \mathbb{R}^-$. Moreover, one checks that $E(z)$ is also analytic across $(-\delta, 0)$. Finally, since $\phi_{1,3}(z)^{1/4} = \mathcal{O}(z^{1/4})$ as $z \rightarrow 0$ it then follows from (5.47) and (5.64) that

$$E(z) = \mathcal{O}(z^{-1/2}), \quad z \rightarrow 0, \quad (5.72)$$

so $E(z)$ cannot have a pole at zero. We conclude that $E(z)$ is analytic in the disk $D(0, \delta)$. By virtue of (5.67), the matching condition (5.66) in the RH problem for $P^{(0)}$ is satisfied.

If $-1 < \nu < 0$, we cannot simply ignore the jumps on the real axis. The local parametrix $P^{(0)}$ is then constructed in the following form

$$\begin{aligned} P^{(0)}(z) = \hat{P}^{(0)}(z) \operatorname{diag}(e^{-n\tilde{\lambda}_1(z)}, e^{-n\tilde{\lambda}_2(z)}, e^{-n\tilde{\lambda}_3(z)}, e^{-n\tilde{\lambda}_4(z)}) Q(z) \\ \times \operatorname{diag}(e^{n\tilde{\lambda}_1(z)}, e^{n\tilde{\lambda}_2(z)}, e^{n\tilde{\lambda}_3(z)}, e^{n\tilde{\lambda}_4(z)}), \end{aligned} \quad (5.73)$$

where $\hat{P}^{(0)}$ is the parametrix for the case $\nu \geq 0$ given in (5.70), and $Q(z)$ is a piecewise constant matrix. More precisely, following the idea in [17, Section 5.6.3], we have

$$Q(z) = I - \frac{e^{-i\pi\nu}}{2i \sin(\pi\nu)} E_{2,3}, \quad (5.74)$$

for z in the region bounded by $(0, \delta)$ and the upper lip of \tilde{L}_1 ,

$$Q(z) = I - \frac{e^{i\pi\nu}}{2i \sin(\pi\nu)} E_{2,3}, \quad (5.75)$$

for z in the region bounded by $(0, \delta)$ and the lower lip of \tilde{L}_1 ,

$$Q(z) = \begin{pmatrix} 1 & 0 & \frac{1}{2i \sin(\pi\nu)} & \frac{e^{\nu\pi i}}{2i \sin(\pi\nu)} \\ 0 & 1 & -\frac{e^{-\nu\pi i}}{2i \sin(\pi\nu)} & -\frac{1}{2i \sin(\pi\nu)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (5.76)$$

for z in the region bounded by $(-\delta, 0)$ and the upper lip of \tilde{L}_1 ,

$$Q(z) = \begin{pmatrix} 1 & 0 & -\frac{1}{2i \sin(\pi\nu)} & \frac{e^{-\nu\pi i}}{2i \sin(\pi\nu)} \\ 0 & 1 & -\frac{e^{\nu\pi i}}{2i \sin(\pi\nu)} & \frac{1}{2i \sin(\pi\nu)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (5.77)$$

for z in the region bounded by $(-\delta, 0)$ and the lower lip of \tilde{L}_1 . With Q given in (5.74)–(5.77), one can check that $P^{(0)}$ defined in (5.73) indeed satisfies the items (1)–(4) in RH problem 5.19 if $-1 < \nu < 0$. For the jump condition (2) we use that $n \equiv 0 \pmod{3}$. For the matching condition (4) we also need the inequalities $\operatorname{Re} \tilde{\lambda}_{3,4}(z) < \operatorname{Re} \tilde{\lambda}_{1,2}(z)$ for z in a neighborhood of the origin, see Lemma 5.4 and (5.12)–(5.13). Moreover, this construction actually works as long as $\sin(\pi\nu) \neq 0$, i.e., $\nu \notin \mathbb{N} \cup \{0\}$.

For **Case II**, the RH problem for $P^{(0)}$ is exactly the same as the one considered in [17, Section 5.6]. Also the construction of $P^{(0)}$ in **Case IV** is similar to Case II. We thus omit the details for these cases.

For **Case III**, the Bessel parametrix will appear in the middle block. More precisely, by setting

$$\phi_2(z) := \left(\tilde{\lambda}_3(z) - \tilde{\lambda}_2(z) \pm \frac{4\pi i}{3} \right)^2 = -c_2^2 z + \mathcal{O}(z^2), \quad \pm \operatorname{Im} z > 0, \quad (5.78)$$

we have

$$\begin{aligned} \hat{P}^{(0)}(z) &= E(z) \operatorname{diag} \left(z^{-\nu/2}, \Psi^{\text{Bessel}} \left(\frac{n^2 \phi_2(z)}{16} \right), z^{\nu/2} \right) \\ &\quad \times \operatorname{diag} \left(1, \sigma_3 e^{\pm \frac{\nu\pi i}{2} \sigma_3} e^{\frac{n}{2}(\tilde{\lambda}_2(z) - \tilde{\lambda}_3(z)) \sigma_3}, 1 \right), \quad \pm \operatorname{Im} z > 0, \end{aligned} \quad (5.79)$$

and the analytic prefactor E is given by

$$\begin{aligned} E(z) &= P^{(\infty)}(z) \operatorname{diag} \left(1, \sigma_3 e^{\mp \frac{\nu\pi i}{2} \sigma_3}, 1 \right) \\ &\quad \times \operatorname{diag} \left(z^{\nu/2}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \left(\frac{n\pi}{2} \phi_2^{1/2} \right)^{\sigma_3/2}, z^{-\nu/2} \right), \quad \pm \operatorname{Im} z > 0. \end{aligned} \quad (5.80)$$

This describes the parametrix if $\nu \geq 0$. If $-1 < \nu < 0$, we define $P^{(0)}(z)$ by (5.73), (5.79), where now $Q(z)$ is a piecewise constant matrix given by

$$Q(z) = I + \frac{i}{2 \sin(\pi\nu)} (E_{2,1} + E_{4,3}), \quad (5.81)$$

for z in the region outside the lens,

$$Q(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{i}{2\sin(\pi\nu)} & 1 & 0 & 0 \\ \frac{i}{2\sin(\pi\nu)} & 0 & 1 & 0 \\ 0 & -\frac{i}{2\sin(\pi\nu)} & \frac{i}{2\sin(\pi\nu)} & 1 \end{pmatrix}, \quad (5.82)$$

for z in the region bounded by $(0, \delta)$ and the upper lip of \tilde{L}_2 , and

$$Q(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{i}{2\sin(\pi\nu)} & 1 & 0 & 0 \\ -\frac{i}{2\sin(\pi\nu)} & 0 & 1 & 0 \\ 0 & \frac{i}{2\sin(\pi\nu)} & \frac{i}{2\sin(\pi\nu)} & 1 \end{pmatrix}, \quad (5.83)$$

for z in the region bounded by $(0, \delta)$ and the lower lip of \tilde{L}_2 . To check the matching condition (4) in RH problem 5.19 we need the inequalities $\operatorname{Re} \tilde{\lambda}_1(z) < \operatorname{Re} \tilde{\lambda}_{2,3}(z) < \operatorname{Re} \tilde{\lambda}_4(z)$ for z in a neighborhood of the origin, see Lemma 5.4 and (5.14).

Finally we build the local parametrix in **Case V**. The construction is much simpler in this case. We set

$$\hat{P}^{(0)}(z) = P^{(\infty)}(z). \quad (5.84)$$

This describes the parametrix if $\nu \geq 0$. If $-1 < \nu < 0$, we define $P^{(0)}(z)$ by (5.73), (5.84), where now

$$Q(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{i}{2\sin(\pi\nu)} & 1 & \frac{ie^{\mp\nu\pi i}}{2\sin(\pi\nu)} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{i}{2\sin(\pi\nu)} & 1 \end{pmatrix}, \quad (5.85)$$

if $\pm \operatorname{Im} z > 0$. One easily checks that conditions (1)–(4) in RH problem 5.19 are satisfied.

Finally, by tracing back the transformation (5.60), we obtain the local approximation $S^{(0)}$ for S from $P^{(0)}$.

5.7 Final transformation

Using the local parametrices described in Sections 5.5 and 5.6, and the global parametrix $S^{(\infty)}$, we define the final transformation as follows

$$R(z) = \begin{cases} S(z)(S^{\text{Airy}}(z))^{-1}, & \text{in the disks around } \{a_j, b_j, c_1, -c_2, c_3\} \setminus \{0\}, \\ S(z)(S^{(0)}(z))^{-1}, & \text{in the disk } D(0, \delta) \text{ around the origin,} \\ S(z)(S^{(\infty)}(z))^{-1}, & \text{elsewhere,} \end{cases} \quad (5.86)$$

where S , S^{Airy} , $S^{(0)}$, and $S^{(\infty)}$ are respectively defined in (5.29); Section 5.5; (5.60), (5.70), (5.73), (5.79), and (5.84); and (5.48).

From our construction of the parametrices, it follows that R satisfies the following RH problem.

- (1) R is analytic in $\mathbb{C} \setminus \Sigma_R$, where the contour Σ_R depends on whether $\nu < 0$ or $\nu \geq 0$ and is different for all five cases.
- (2) R has jumps $R_+ = R_- J_R$ on Σ_R that satisfy

$$J_R(z) = I + \mathcal{O}(1/n), \quad (5.87)$$

uniformly for z on the boundaries of the disks;

$$J_R(x) = I + \mathcal{O}(x^\nu e^{-cn}), \quad (5.88)$$

on $(-\delta, 0)$ (Case I), or on $(0, \delta)$ (Case III), or on $(-\delta, 0) \cup (0, \delta)$ (Cases II, IV and V), for some constant $c > 0$, if $\nu \geq 0$; and

$$J_R(z) = I + \mathcal{O}(e^{-cn|z|}), \quad (5.89)$$

on the other parts of Σ_R , for some constant $c > 0$.

(3) $R(z) = I + \mathcal{O}(1/z)$ as $z \rightarrow \infty$.

In case $\nu \geq 0$, the estimate (5.88) is not trivial. We prove it in Case I. In that case $R(z)$ is clearly analytic in $D(0, \delta) \setminus (-\delta, 0]$ with the following jump on $(-\delta, 0)$

$$\begin{aligned} R_-(x)^{-1} R_+(x) &= P_+^{(0)}(-x) J_P(-x)^{-1} P_-^{(0)}(-x)^{-1}, \\ &= P_+^{(0)}(-x) \left(I - e^{n(\tilde{\lambda}_3, -\tilde{\lambda}_{2,+})(-x)} E_{2,3} \right) P_-^{(0)}(-x)^{-1}, \\ &= I - e^{n(\tilde{\lambda}_3, -\tilde{\lambda}_{2,+})(-x)} P^{(0)}(-x) E_{2,3} P^{(0)}(-x)^{-1}. \end{aligned}$$

Now for $\nu \geq 0$ the matrix $P^{(0)}(-x) E_{2,3} P^{(0)}(-x)^{-1} = \mathcal{O}(|x|^\nu)$ as $x \rightarrow 0$, $x < 0$. Indeed (following the proof of [17, Lemma 5.6]) we observe that

$$P^{(0)}(-x) E_{2,3} P^{(0)}(-x)^{-1} = P^{(0)}(-x) \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} P^{(0)}(-x)^{-1}.$$

Then if $\nu > 0$ we find from (5.65) and the fact that $\det P^{(0)}(z) \equiv 1$ that both factors

$$P^{(0)}(-x) \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} P^{(0)}(-x)^{-1}$$

behave like $\mathcal{O}(|x|^{\nu/2})$ as $x \rightarrow 0$, $x < 0$, which proves the statement for $\nu > 0$. In case $\nu = 0$, this approach does not work as it would lead to a bound $\mathcal{O}(\log |x|)$. However, we do find that the first factor $P^{(0)}(-x) \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T$ remains bounded as $x \rightarrow 0$, $x < 0$. For the second factor we observe from (5.68) and $\det \Psi^{\text{Bessel}}(\zeta) \equiv 1$ that

$$\left(\Psi^{\text{Bessel}} \right)^{-1}(\zeta) = \begin{pmatrix} \mathcal{O}(\log |\zeta|) & \mathcal{O}(\log |\zeta|) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, \quad \text{as } \zeta \rightarrow 0.$$

Combined with (5.70) and the fact that $E(z)$ is bounded near the origin we obtain that $\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} P^{(0)}(-x)^{-1}$ is also bounded as $x \rightarrow 0$, $x < 0$, which proves the statement in case $\nu = 0$. Cases II, III, IV, and V can be similarly treated.

Then from standard arguments we may conclude that

$$R(z) = I + \mathcal{O}\left(\frac{1}{n(|z|+1)}\right), \quad (5.90)$$

as $n \rightarrow \infty$, uniformly for z in the complex plane outside of Σ_R .

5.8 Proof of Theorem 2.6

We follow the approach in [23, Section 9.3] and start the proof with a lemma.

Lemma 5.21. *For every $x \in S(\mu_1) \setminus \bigcup_{k=1}^N (D(a_k, \epsilon) \cup D(b_k, \epsilon))$ we have*

$$S_+^{-1}(y) S_+(x) = \begin{pmatrix} I_2 + \mathcal{O}(x-y) & * \\ * & * \end{pmatrix}, \quad \text{as } y \rightarrow x,$$

uniformly in n . The $*$ entries denote unimportant 2×2 blocks.

Proof. The proof is standard, see e.g. [23, Lemma 9.9]. □

Recall the formula (2.11) that expresses K_n in terms of Y . The idea is then to write this expression in terms of R instead of Y by applying all respective transformations

$$X \mapsto U \mapsto T \mapsto S \mapsto R,$$

introduced in the steepest descent analysis. Meanwhile, we let n tend to infinity, so that we can exploit the conclusion of the steepest descent analysis (5.90).

Let $x, y > 0$. First, unfolding the transformation $Y \mapsto X$ given in (5.4) we get

$$K_n(x, y) = \frac{y^{\nu/2} x^{-\nu/2}}{2\pi i(x-y)} \begin{pmatrix} 0 & w_{0,n}(y) & w_{1,n}(y) & w_{2,n}(y) \end{pmatrix} \\ \times \text{diag} \left(1, D(z)^{-1} W_{n,+}^{-T}(y) e^{n\Theta_+(y)} \right) X_+^{-1}(y) X_+(x) \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T.$$

Using (5.3) this boils down to

$$K_n(x, y) = \frac{y^{\nu/2} x^{-\nu/2} e^{n(\theta_{1,+}(y) - V(y))}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} X_+^{-1}(y) X_+(x) \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T. \quad (5.91)$$

Next, by the transformation $X \mapsto U$ described by (5.19) we obtain

$$e^{n(V(y) - V(x))} K_n(x, y) = y^{\nu/2} x^{-\nu/2} \frac{e^{n(\lambda_{2,+}(y) - \lambda_{1,+}(x))}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} U_+^{-1}(y) U_+(x) \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T.$$

The opening of global lenses $U \mapsto T$ in (5.28) does not effect the above expression. The opening of the local lens $T \mapsto S$ in (5.29), however, does have impact

$$e^{n(V(y) - V(x))} K_n(x, y) = \frac{y^{\nu/2} x^{-\nu/2}}{2\pi i(x-y)} \\ \times \begin{pmatrix} -\chi_{S(\mu_1)}(y) e^{n\lambda_{1,+}(y)} & e^{n\lambda_{2,+}(y)} & 0 & 0 \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-n\lambda_{1,+}(x)} & \chi_{S(\mu_1)}(x) e^{-n\lambda_{2,+}(x)} & 0 & 0 \end{pmatrix}^T,$$

where $\chi_{S(\mu_1)}$ denotes the characteristic function of the set $S(\mu_1)$.

Let $x \in S(\mu_1) \setminus \{a_k, b_k \mid k = 1, \dots, N\}$. The factors $e^{n(V(y) - V(x))}$ and $y^{\nu/2} x^{-\nu/2}$ disappear as $y \rightarrow x$. Then Lemma 5.21 yields

$$K_n(x, x) = \lim_{y \rightarrow x} \frac{e^{n(\lambda_{2,+}(y) - \lambda_{2,+}(x))} - e^{n(\lambda_{1,+}(y) - \lambda_{1,+}(x))}}{2\pi i(x-y)} + \mathcal{O}(1), \\ = \frac{n}{2\pi i} \frac{d}{dx} (\lambda_{1,+}(x) - \lambda_{2,+}(x)) + \mathcal{O}(1).$$

Hence,

$$\lim_{n \rightarrow \infty} \frac{1}{n} K_n(x, x) = \frac{1}{2\pi i} \frac{d}{dx} (\lambda_{1,+}(x) - \lambda_{2,+}(x)) \\ = \frac{1}{\pi i} \frac{d}{dx} (\lambda_{1,+}^{\text{NC}}(\sqrt{x}) - \lambda_{2,+}^{\text{NC}}(\sqrt{x})) \\ = \frac{\rho_1^{\text{NC}}(\sqrt{x})}{\sqrt{x}} \\ = \rho_1(x),$$

where the second equality follows from (5.15), the third is taken from [23, p. 116], and the last one is (2.23).

6 Triple scaling limit in the quadratic/linear case

In this part we study the very concrete case of the chiral two-matrix model with potentials

$$V(x) = x, \quad W(y) = \frac{y^2}{2} + \alpha y.$$

For this case we were able to construct a phase diagram in the (α, τ) -plane, see Figure 1. Very remarkable is the occurrence of a multi-critical point for the parameter values $\alpha = -1, \tau = 1$. In this part we will study a triple scaling limit to this point leading to the chiral version of the main result in [20]. An essential point in the proof is the construction of the local parametrix around zero. In this construction we will make use of the solution to a model RH problem introduced in [16].

The proof of Theorem 2.8 is again based on a steepest descent analysis. This analysis is very similar to the one performed for the non-critical cases. In fact the first four transformations are almost exactly equal. The only difference is that we will need to modify the λ -functions, which will be introduced first.

6.1 Modified λ -functions

Let us first introduce an auxiliary parameter γ that is completely determined by α and τ but will prove to be convenient for notation. We define $\gamma = \gamma(\alpha, \tau)$ as the solution of

$$\alpha\tau^{2/3} = \frac{3}{\gamma} - 9\gamma^2 + 5\tau^{4/3}\gamma, \quad (6.1)$$

that tends to 1 as $\tau \rightarrow 1$ and $\alpha \rightarrow -1$. In the triple scaling limit, i.e. we let α and τ depend on n as in (2.25) while $n \rightarrow \infty$, we have

$$\gamma = 1 + \frac{1}{3}an^{-1/3} + \left(\frac{11}{144}a^2 + \frac{47}{48}b\right)n^{-2/3} + \mathcal{O}(n^{-1}). \quad (6.2)$$

Lemma 6.1. *There exist functions λ_j , $j = 1, 2, 3, 4$, analytic on $\mathbb{C} \setminus \mathbb{R}$ that satisfy the following conditions.*

(a) *As $z \rightarrow \infty$ we have*

$$\lambda_1(z) = z - \log(z) + \ell_1 + \mathcal{O}(z^{-1}), \quad (6.3)$$

$$\lambda_2(z) = \theta_1(z) + \frac{1}{3} \log z + \ell_2 + Cz^{-1/3} + Dz^{-2/3} + \mathcal{O}(z^{-1}), \quad (6.4)$$

$$\lambda_3(z) = \theta_2(z) + \begin{cases} \frac{1}{3} \log z + \ell_3 + C\omega z^{-1/3} + D\omega^2 z^{-2/3} + \mathcal{O}(z^{-1}) & \text{in } \mathbb{C}_+, \\ \frac{1}{3} \log z + \ell_4 + C\omega^2 z^{-1/3} + D\omega z^{-2/3} + \mathcal{O}(z^{-1}) & \text{in } \mathbb{C}_-, \end{cases} \quad (6.5)$$

$$\lambda_4(z) = \theta_3(z) + \begin{cases} \frac{1}{3} \log z + \ell_4 + C\omega^2 z^{-1/3} + D\omega z^{-2/3} + \mathcal{O}(z^{-1}) & \text{in } \mathbb{C}_+, \\ \frac{1}{3} \log z + \ell_3 + C\omega z^{-1/3} + D\omega^2 z^{-2/3} + \mathcal{O}(z^{-1}) & \text{in } \mathbb{C}_-, \end{cases} \quad (6.6)$$

where $C, D \in \mathbb{R}$ and ℓ_j , $j = 1, 2, 3, 4$, are constants satisfying

$$\ell_3 - \ell_2 = \ell_2 - \ell_4 = \frac{4}{3}\pi i. \quad (6.7)$$

(b) *There exists a constant $c > 0$ such that the modified λ -functions satisfy the following jump conditions.*

(i) *On \mathbb{R}^+ we have*

$$\lambda_{1,\pm} = \lambda_{2,\mp} \quad \text{on } (0, c), \quad (6.8)$$

$$\lambda_{j,+} = \lambda_{j,-} \quad \text{on } (c, \infty), \quad j = 1, 2, \quad (6.9)$$

$$\lambda_{3,\pm} = \lambda_{4,\mp} \quad \text{on } (0, \infty). \quad (6.10)$$

(ii) On \mathbb{R}^- we have

$$\lambda_{j,+} = \lambda_{j,-} - 2\pi i, \quad j = 1, 4, \quad (6.11)$$

$$\lambda_{2,\pm} = \lambda_{3,\mp} \pm 2\pi i. \quad (6.12)$$

(c) In a neighborhood of the origin we have

$$\begin{aligned} \lambda_1(z) &= zK(z) + \begin{cases} F(z)z^{1/4} + G(z)z^{1/2} + H(z)z^{3/4} & \text{in } \mathbb{C}_+, \\ iF(z)z^{1/4} - G(z)z^{1/2} - iH(z)z^{3/4} + 2\pi i & \text{in } \mathbb{C}_-, \end{cases} \\ \lambda_2(z) &= zK(z) + \begin{cases} iF(z)z^{1/4} - G(z)z^{1/2} - iH(z)z^{3/4} + 2\pi i & \text{in } \mathbb{C}_+, \\ F(z)z^{1/4} + G(z)z^{1/2} + H(z)z^{3/4} & \text{in } \mathbb{C}_-, \end{cases} \\ \lambda_3(z) &= zK(z) + \begin{cases} -iF(z)z^{1/4} - G(z)z^{1/2} + iH(z)z^{3/4} + 2\pi i & \text{in } \mathbb{C}_+, \\ -F(z)z^{1/4} + G(z)z^{1/2} - H(z)z^{3/4} & \text{in } \mathbb{C}_-, \end{cases} \\ \lambda_4(z) &= zK(z) + \begin{cases} -F(z)z^{1/4} + G(z)z^{1/2} - H(z)z^{3/4} & \text{in } \mathbb{C}_+, \\ -iF(z)z^{1/4} - G(z)z^{1/2} + iH(z)z^{3/4} + 2\pi i & \text{in } \mathbb{C}_-. \end{cases} \end{aligned}$$

where F, G, H, K are analytic functions satisfying

$$F(0) = 4e^{3\pi i/4}\gamma^{1/4} \left(-2\gamma^2 + \frac{1}{\gamma} + \tau^{4/3}\gamma \right), \quad (6.13)$$

$$G(0) = 2i\gamma^{-1/2} \left(\frac{3}{2}\gamma^2 - \frac{1}{4\gamma} - \frac{5}{4}\tau^{4/3}\gamma \right), \quad (6.14)$$

$$H(0) = 2e^{\pi i/4}\gamma^{-5/4} \left(\frac{1}{2}\gamma^2 - \frac{1}{12\gamma} + \frac{1}{4}\tau^{4/3}\gamma \right). \quad (6.15)$$

Proof. Define

$$\lambda_j(z) = 2\lambda_j^{\text{NC}}(\sqrt{z}), \quad j = 1, 2, 3, 4, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (6.16)$$

where λ_j^{NC} denote the modified λ -functions introduced in [20, Section 3.4]. Then (a) follows from [20, Lemma 3.11]. (b)(i) is direct from [20, Lemma 3.10]. To prove (ii) we need some symmetry conditions on λ_j^{NC} . By [20, (3.15), (3.23)–(3.24), (3.29)–(3.30)], it follows that for $x > 0$

$$\begin{aligned} \lambda_j^{\text{NC}}(ix) &= \lambda_j^{\text{NC}}(-ix) - \pi i, & j = 1, 4, \\ \lambda_{j,\mp}^{\text{NC}}(ix) &= \lambda_{j,\pm}^{\text{NC}}(-ix) + \pi i, & j = 2, 3. \end{aligned}$$

This, together with (6.16), implies (b)(ii), where we take $c = (c^{\text{NC}})^2$. Finally (c) follows from [20, Lemma 3.12] where we put

$$\begin{cases} F(z) = 2F^{\text{NC}}(\sqrt{z}), & G(z) = 2G^{\text{NC}}(\sqrt{z}), \\ H(z) = 2H^{\text{NC}}(\sqrt{z}), & K(z) = 2K^{\text{NC}}(\sqrt{z}), \end{cases} \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

□

Remark 6.2. The constants γ, c , etc. and the functions F, G, H, K, λ_j , etc. all implicitly depend on α and τ (and thus also on n via the triple scaling limit). When dealing with these functions associated with the critical values of the parameters $\alpha = -1, \tau = 1$, we add a star to the notation. Thus, we write $\gamma^*, c^*, F^*, G^*, H^*, K^*, \lambda_j^* \dots$

6.2 The transformations $Y \mapsto X \mapsto U \mapsto T \mapsto S$

The first transformations $Y \mapsto X \mapsto U \mapsto T \mapsto S$ of the steepest descent analysis in the critical case are almost the same as before. The main difference is that we use the modified lambda functions rather than the ordinary ones. Apart from that also some details have to be changed. We list them here.

Transformation $Y \mapsto X$.

This transformation is exactly as in Section 5.1.

Transformation $X \mapsto U$.

Definition 5.7 of $U(z)$ has to be slightly changed. Besides the fact that we replace the λ -functions by the modified λ -functions, we also have to change the definition of the diagonal matrix L by

$$L(z) = \begin{cases} -\text{diag}(\ell_1, \ell_2, \ell_3, \ell_4) & \text{for } \text{Im } z > 0, \\ -\text{diag}(\ell_1, \ell_2, \ell_4, \ell_3) & \text{for } \text{Im } z < 0. \end{cases} \quad (6.17)$$

Then, under the assumption that $n \equiv 0 \pmod{3}$, U solves RH problem 5.8 if we put $N := 1$, $a_1 := 0$, $b_1 := c$, and $c_2 = c_3 := 0$.

Transformations $U \mapsto T \mapsto S$: opening of lenses.

Here, we open the local lens L_1 around $[0, c]$ and unbounded lenses L_2 around \mathbb{R}^- and L_3 around \mathbb{R}^+ . We want to do this such that the off-diagonal entries of the jump matrices on the lips of these lenses tend exponentially fast to zero as $n \rightarrow \infty$. This is only possible outside a shrinking disk around the origin. It will be sufficient for our purposes to let this disk shrink with speed $n^{-1/3}$

$$D(0, \rho n^{-1/3}) = \{z \in \mathbb{C} \mid |z| < \rho n^{-1/3}\},$$

where $\rho > 0$ is a constant that will be chosen sufficiently small later on.

Lemma 6.3. *The lenses L_j , $j = 1, 2, 3$, can be opened (independently of n) such that*

$$\text{Re}(\lambda_1(z) - \lambda_2(z)) \leq -dn^{-1/2}, \quad \text{for } z \text{ on the lips of } L_1 \text{ but outside } D(0, \rho n^{-1/3}), \quad (6.18)$$

$$\text{Re}(\lambda_3(z) - \lambda_2(z)) \leq -dn^{-1/2} \max(1, |z|^{2/3}), \quad \text{for } z \text{ on the lips of } L_2 \text{ but outside } D(0, \rho n^{-1/3}), \quad (6.19)$$

$$\text{Re}(\lambda_3(z) - \lambda_4(z)) \leq -dn^{-1/2} \max(1, |z|^{2/3}), \quad \text{for } z \text{ on the lips of } L_3 \text{ but outside } D(0, \rho n^{-1/3}), \quad (6.20)$$

for sufficiently large n and a fixed constant $d > 0$. Moreover, there exists a constant d' such that

$$\text{Re}(\lambda_2(x) - \lambda_1(x)) \leq -d'x, \quad x \in (c^* + \epsilon, \infty), \quad (6.21)$$

where $\epsilon > 0$ is a small number.

Proof. Estimates (6.18)–(6.20) are immediate from (6.16) and [20, Lemmas 4.6 and 4.9] where we define $L_j = \left(L_j^{\text{NC}}\right)^2$. Estimate (6.21) follows in the same way from the following result that holds in the context of [20] but was not explicitly mentioned there: there exists $d > 0$ such that for sufficiently large n

$$\text{Re}(\lambda_2^{\text{NC}}(x) - \lambda_1^{\text{NC}}(x)) \leq -dx^2, \quad x \in (-\infty, -(c^*)^{\text{NC}} - \epsilon) \cup ((c^*)^{\text{NC}} + \epsilon, \infty).$$

□

In this way we arrive at the following RH problem for S .

RH problem 6.4. *The matrix-valued function S is the unique solution of the following RH problem*

- (1) $S(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma_S$, where Σ_S is the contour consisting of the real axis and the lips of the lenses L_i , $i = 1, 2, 3$. These lenses are chosen such that the estimates in Lemma 6.3 hold.
- (2) For $z \in \Sigma_S$, S has a jump

$$S_+(z) = S_-(z) \begin{cases} \text{diag}((J_S)_1(z), (J_S)_3(z)), & \text{for } z \text{ in } \mathbb{R}^+ \text{ and the lips of } L_1, L_3, \\ \text{diag}(e^{\nu\pi i}, (J_S)_2(z), e^{-\nu\pi i}), & \text{for } z \text{ in } \mathbb{R}^- \text{ and the lips of } L_2, \end{cases}$$

where

$$(J_S)_1 = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (0, c), \\ \begin{pmatrix} 1 & e^{n(\lambda_2, + - \lambda_1, -)} \\ 0 & 1 \end{pmatrix}, & \text{on } (c, \infty), \\ \begin{pmatrix} 1 & 0 \\ e^{n(\lambda_1 - \lambda_2)} & 1 \end{pmatrix}, & \text{on the lips of } L_1, \\ I_2, & \text{on the lips of } L_3, \end{cases}$$

$$(J_S)_2 = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (-\infty, 0), \\ \begin{pmatrix} 1 & -e^{\pm\nu\pi i} e^{n(\lambda_3 - \lambda_2)} \\ 0 & 1 \end{pmatrix}, & \text{on the upper/lower lip of } L_2, \end{cases}$$

and

$$(J_S)_3 = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (0, +\infty), \\ I_2, & \text{on the lips of } L_1, \\ \begin{pmatrix} 1 & 0 \\ e^{n(\lambda_3 - \lambda_4)} & 1 \end{pmatrix}, & \text{on the lips of } L_3. \end{cases}$$

- (3) As $z \rightarrow \infty$ with $\pm \text{Im } z > 0$, we have

$$S(z) = [I + \mathcal{O}(z^{-1})] \text{diag}(1, z^{1/3}, z^{-1/3}, 1) \text{diag}(z^{\nu/2}, z^{-\nu/6} A_{\pm}^{-T}) \text{diag}(1, 1, \sigma^{\pm}, \sigma^{\mp 1}).$$

- (4) $S(z)$ has the same behavior near the origin as $X(z)$, see (5.6), provided that $z \rightarrow 0$ outside the lenses that end in 0.

We will construct local and global parametrices for $P(z)$ as described in Section 5.6, i.e. $P(z)$ is established from the solution $S(z)$ to RH problem 6.4 by the transformation (5.60). More precisely, we have

RH problem 6.5. *The matrix-valued function P satisfies the following RH problem.*

- (1) $P(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma_P$, where Σ_P is the contour consisting of the real axis and the lips of the lenses $\tilde{L}_i = -L_i$, $i = 1, 2, 3$.
- (2) For $z \in \Sigma_P$, P has a jump

$$P_+(z) = P_-(z) \begin{cases} \text{diag}(1, (J_P(z))_2, 1), & \text{for } z \text{ in } \mathbb{R}^+ \text{ and the lips of } \tilde{L}_2, \\ \text{diag}((J_P(z))_1, (J_P(z))_3), & \text{for } z \text{ in } \mathbb{R}^- \text{ and the lips of } \tilde{L}_1, \tilde{L}_3, \end{cases}$$

where

$$(J_P)_1 = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (-\infty, 0), \\ I_2, & \text{on the lips of } \tilde{L}_1, \\ \begin{pmatrix} 1 & -e^{\pm\nu\pi i}e^{n(\tilde{\lambda}_2-\tilde{\lambda}_1)} \\ 0 & 1 \end{pmatrix}, & \text{on the upper/lower lip of } \tilde{L}_3, \end{cases}$$

$$(J_P)_2 = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (0, \infty), \\ \begin{pmatrix} 1 & 0 \\ e^{n(\tilde{\lambda}_2-\tilde{\lambda}_3)} & 1 \end{pmatrix}, & \text{on the lips of } \tilde{L}_2, \end{cases}$$

and

$$(J_P)_3 = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (-c, 0), \\ \begin{pmatrix} e^{-\nu\pi i} & 0 \\ -e^{n(\tilde{\lambda}_3, -\tilde{\lambda}_{4,+})} & e^{\nu\pi i} \end{pmatrix}, & \text{on } (-\infty, -c), \\ \begin{pmatrix} 1 & -e^{\pm\nu\pi i+n(\tilde{\lambda}_4-\tilde{\lambda}_3)} \\ 0 & 1 \end{pmatrix}, & \text{on the upper/lower lip of } \tilde{L}_1, \\ I_2, & \text{on the lips of } \tilde{L}_3, \end{cases}$$

where

$$\tilde{\lambda}_i(z) = \lambda_{5-i}(-z), \quad i = 1, 2, 3, 4. \quad (6.22)$$

(3) If $z \rightarrow 0$ outside the lenses that end in 0, we have

$$\begin{cases} P(z) \operatorname{diag}(|z|^{\nu/2}, |z|^{-\nu/2}, |z|^{\nu/2}, |z|^{-\nu/2}) = \mathcal{O}(1), & \text{if } \nu > 0, \\ P(z) \operatorname{diag}((\log |z|)^{-1}, 1, (\log |z|)^{-1}, 1) = \mathcal{O}(1), & \text{if } \nu = 0, \\ P(z) = \mathcal{O}(|z|^{\nu/2}), \quad P^{-1}(z) = \mathcal{O}(|z|^{\nu/2}), & \text{if } -1 < \nu < 0. \end{cases} \quad (6.23)$$

6.3 Global parametrix

By ignoring all the exponentially small terms in the RH problem for P , we obtain a RH problem for $P^{(\infty)}$. This global parametrix $P^{(\infty)}$ can be constructed by first constructing $S^{(\infty)}$ along the lines of Section 5.4 and then again applying the transformation $S^{(\infty)} \mapsto P^{(\infty)}$ as given by (5.64). Its behavior around the origin is given by

$$\begin{cases} P^{(\infty)}(z) = \hat{P}_{\pm} z^{-3/8} + \mathcal{O}(z^{-1/8}), \\ (P^{(\infty)})^{-1}(z) = \hat{Q}_{\pm} z^{-3/8} + \mathcal{O}(z^{-1/8}), \end{cases} \quad \text{as } z \rightarrow 0, \quad \pm \operatorname{Im} z > 0, \quad (6.24)$$

for constant matrices $\hat{P}_{\pm}, \hat{Q}_{\pm}$. Clearly

$$\hat{P}_{\pm} \hat{Q}_{\pm} = \hat{Q}_{\pm} \hat{P}_{\pm} = 0. \quad (6.25)$$

Furthermore, in view of the jump conditions for $P^{(\infty)}$, it is readily seen that

$$\hat{P}_+ = \hat{P}_- \operatorname{diag} \left(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \quad (6.26)$$

$$\hat{P}_+ = \hat{P}_- e^{3\pi i/4} \operatorname{diag} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right). \quad (6.27)$$

Eliminating \widehat{P}_+ from these two formulas we get

$$\widehat{P}_- = \widehat{P}_- e^{3\pi i/4} \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Iterating this relation we see

$$\widehat{P}_- = \widehat{P}_- \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \quad (6.28)$$

Similarly, we have

$$\widehat{Q}_+ = \text{diag} \left(1, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1 \right) \widehat{Q}_- \quad (6.29)$$

$$\widehat{Q}_+ = e^{3\pi i/4} \text{diag} \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \widehat{Q}_-, \quad (6.30)$$

$$\widehat{Q}_- = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \widehat{Q}_-. \quad (6.31)$$

These relations will be helpful later.

6.4 Local parametrix at c

As in Section 5.5 the local parametrix $S^{(c)}$ around c can be built in the standard way with the help of Airy functions, see e.g. [23]. We omit the details here.

6.5 Local parametrix at the origin

In this section we build a local parametrix $P^{(0)}$ near the origin. Here the analysis is essentially different from the noncritical situation discussed in the previous section.

Transformation of the RH problem for $M(\zeta)$

To build the local parametrix near the origin we will use a slightly modified version of RH problem 2.7. We put $\tilde{\nu} = \nu + 1/2$ and set

$$N(\zeta) = \text{diag} \left(\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, 1, i \right) M(\zeta) \text{diag} \left(\begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}, 1, -i \right). \quad (6.32)$$

The jumps for N are shown in Figure 10.

The asymptotics of N as $\zeta \rightarrow \infty$ is given by

$$N(\zeta) = \left(I + \frac{N_1}{\zeta} + \frac{N_2}{\zeta^2} + \mathcal{O} \left(\frac{1}{\zeta^3} \right) \right) \text{diag}(\zeta^{-1/4}, (-\zeta)^{-1/4}, (-\zeta)^{1/4}, \zeta^{1/4}) \\ \times \tilde{A} \text{diag} \left(e^{-\psi_1(\zeta) - t\zeta}, e^{-\psi_2(\zeta) + t\zeta}, e^{\psi_2(\zeta) + t\zeta}, e^{\psi_1(\zeta) - t\zeta} \right) \quad (6.33)$$

with

$$\tilde{A} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}, \quad (6.34)$$

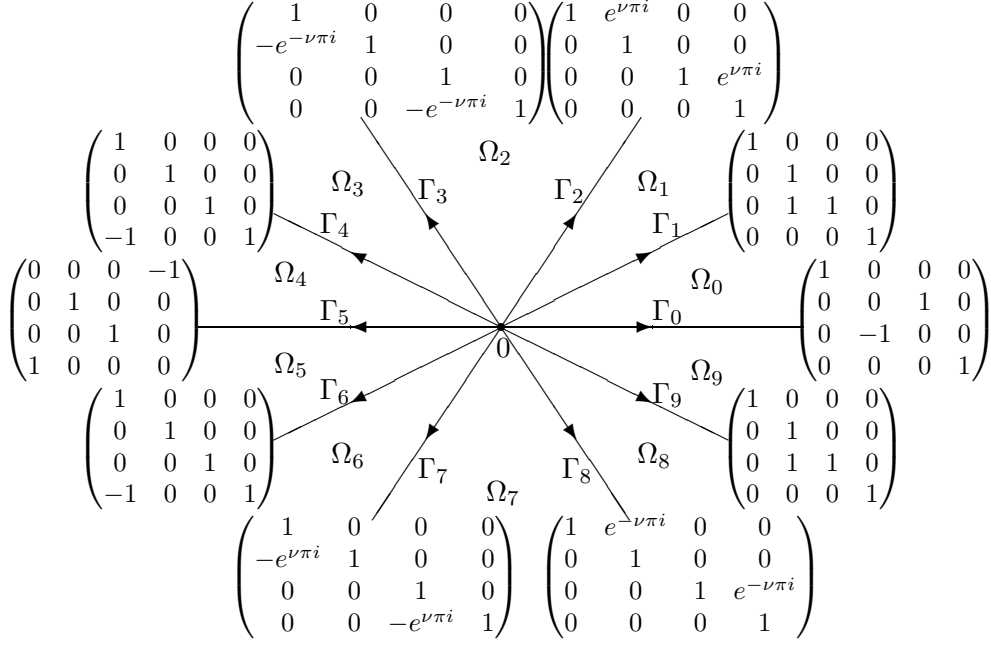


Figure 10: The figure shows the jump matrices J_k , $k = 0, \dots, 9$, in the RH problem for $N = N(\zeta)$.

and the behavior of $N(\zeta)$ for $\zeta \rightarrow 0$ is given by

$$N(\zeta) = \mathcal{O}(\zeta^{\tilde{\nu}}), \quad N^{-1}(\zeta) = \mathcal{O}(\zeta^{\tilde{\nu}}), \quad \text{if } \tilde{\nu} \leq 0,$$

and

$$\begin{cases} N(\zeta) \text{diag}(\zeta^{\tilde{\nu}}, \zeta^{-\tilde{\nu}}, \zeta^{\tilde{\nu}}, \zeta^{-\tilde{\nu}}) = \mathcal{O}(1), & \zeta \in \Omega_1 \cup \Omega_8, \\ N(\zeta) \text{diag}(\zeta^{-\tilde{\nu}}, \zeta^{\tilde{\nu}}, \zeta^{-\tilde{\nu}}, \zeta^{\tilde{\nu}}) = \mathcal{O}(1), & \zeta \in \Omega_3 \cup \Omega_6, \\ N(\zeta) = \mathcal{O}(\zeta^{-\tilde{\nu}}), & \zeta \notin (\Omega_1 \cup \Omega_3 \cup \Omega_6 \cup \Omega_8), \end{cases} \quad \text{if } \tilde{\nu} \geq 0.$$

Note that the behavior at infinity can be rewritten as

$$N(\zeta) = \text{diag}(\zeta^{-1/4}, (-\zeta)^{-1/4}, (-\zeta)^{1/4}, \zeta^{1/4}) \tilde{A} \left(I + \frac{\tilde{N}_{1,\pm}}{\zeta^{1/2}} + \frac{\tilde{N}_{2,\pm}}{\zeta} + \mathcal{O}\left(\frac{1}{\zeta^{3/2}}\right) \right) \times \text{diag}\left(e^{-\psi_1(\zeta)-t\zeta}, e^{-\psi_2(\zeta)+t\zeta}, e^{\psi_2(\zeta)+t\zeta}, e^{\psi_1(\zeta)-t\zeta}\right) \quad (6.35)$$

as $\zeta \rightarrow \infty$ within the upper/lower half plane. Here

$$\tilde{N}_{1,\pm} = \tilde{A}^{-1} \text{diag}\left(1, e^{\mp\pi i/4}, 0, 0\right) N_1 \text{diag}\left(0, 0, e^{\mp\pi i/4}, 1\right) \tilde{A}. \quad (6.36)$$

For further use, we record the symmetry relation (see also [16])

$$N(-\zeta; r_1, r_2, s, t) = \text{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \times N(\zeta; r_2, r_1, s, t) \text{diag}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right). \quad (6.37)$$

Note that the order of r_1 and r_2 differs at both sides of the equality.

A corollary of this relation and (6.35) is

$$\frac{\tilde{N}_{1,\mp}(r_1, r_2, s, t)}{(-\zeta)^{1/2}} = \text{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \times \frac{\tilde{N}_{1,\pm}(r_2, r_1, s, t)}{\zeta^{1/2}} \text{diag}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right), \quad \pm \text{Im } \zeta > 0. \quad (6.38)$$

Construction of the local parametrix

Here we construct the local parametrix $P^{(0)}$ around the origin. We will do this inside a shrinking disk $D(0, \rho n^{-1/3})$, i.e. we want to solve the following RH problem.

RH problem 6.6. *We look for $P^{(0)}$ satisfying the following conditions.*

- (1) $P^{(0)}(z)$ is analytic for $z \in D(0, \rho n^{-1/3}) \setminus \Sigma_P$, where $D(0, \rho n^{-1/3})$ denotes the disk of radius $\rho n^{-1/3}$ around 0.
- (2) $P^{(0)}$ satisfies the jumps

$$P_+^{(0)} = P_-^{(0)} J_P, \quad \text{on } \Sigma_P \cap D(0, \rho n^{-1/3}),$$

where J_P is the jump matrix in RH problem 6.5.

- (3) As $n \rightarrow \infty$, we have that

$$P^{(0)}(z) = \left(I + Z(z) + \mathcal{O}(n^{-1/12}) \right) P^{(\infty)}(z), \quad \text{uniformly for } |z| = \rho n^{-1/3}, \quad (6.39)$$

where

$$Z(z) = \mathcal{O}(1), \quad \text{as } n \rightarrow \infty, \quad \text{uniformly for } |z| = \rho n^{-1/3}. \quad (6.40)$$

We define the local parametrix as follows (compare with [16]):

$$P^{(0)}(z) = E_n(z) N \left(n^{2/3} f(z); r_1(z), r_2(z), n^{2/3} s(z), n^{1/3} t(z) \right) \Lambda(z), \quad (6.41)$$

where

$$\Lambda(z) = \text{diag} \left(e^{n\tilde{\lambda}_1(z)}, \dots, e^{n\tilde{\lambda}_4(z)} \right),$$

and, as before, $\tilde{\lambda}_j(z) = \lambda_{5-j}(-z)$, $j = 1, 2, 3, 4$. The parameters r_1 , r_2 , s , and t depend in a mild way on z and also on n , although this is not indicated in the notation. The prefactor $E_n(z)$ does depend on n and is analytic in a neighborhood of the origin except on the negative real line where it has a jump.

We set

$$f(z) = z^{1/2}, \quad (6.42)$$

where we put the branch cut along $(-\infty, 0]$. Furthermore we define for $z \in D(0, \rho n^{-1/3}) \setminus (-\rho n^{-1/3}, 0]$

$$r_1(z) = \frac{3}{2} e^{-\pi i/4} H(-z) - \frac{3}{2} e^{\pi i/4} (F(-z) - F(0)) z^{-1/2}, \quad (6.43)$$

$$r_2(z) = \frac{3}{2} e^{-\pi i/4} H(-z) + \frac{3}{2} e^{\pi i/4} (F(-z) - F(0)) z^{-1/2}, \quad (6.44)$$

$$s(z) = \frac{1}{2} e^{-3\pi i/4} F(0), \quad (6.45)$$

$$t(z) = iG(-z), \quad (6.46)$$

where F , G , and H are the analytic functions introduced in item (c) of Lemma 6.1. Clearly $t(z)$ and $s(z)$ are analytic in the disk $D(0, \rho n^{-1/3})$ for n large enough. $r_1(z)$ and $r_2(z)$ are only analytic in $D(0, \rho n^{-1/3}) \setminus (-\rho n^{-1/3}, 0]$, but they are each others analytic continuation across the interval $(-\rho n^{-1/3}, 0)$.

In the construction of the parametrix, apart from the analytic structure of the parameters $r_1(z)$, $r_2(z)$, $s(z)$, and $t(z)$, we will also need their asymptotic behavior as $n \rightarrow \infty$.

Lemma 6.7. *The functions $s(z)$ and $t(z)$ are analytic for $z \in D(0, \rho n^{-1/3})$ and n sufficiently large. The functions $r_1(z)$ and $r_2(z)$ are analytic in $D(0, \rho n^{-1/3}) \setminus (-\rho n^{-1/3}, 0]$ and satisfy*

$$r_{1,\pm}(x) = r_{2,\mp}(x), \quad -\rho n^{-1/3} < x < 0. \quad (6.47)$$

Moreover, there exists a constant $\hat{G} \in \mathbb{C}$, independent of z and n , such that

$$\begin{cases} \lim_{n \rightarrow \infty} r_j(n^{-1/3}z) = 2, \\ \lim_{n \rightarrow \infty} n^{2/3}s(n^{-1/3}z) = \frac{1}{2}(a^2 - 5b), \\ \lim_{n \rightarrow \infty} n^{1/3}t(n^{-1/3}z) = 2a + \hat{G}z, \end{cases} \quad (6.48)$$

for $|z| < \rho$ and $j = 1, 2$.

Proof. The analytic structure of these functions for sufficiently large n was already discussed. The limiting behavior for r_j is obvious. To obtain the limiting behavior for t we write

$$\begin{cases} G(z) = G(0) + G_1 z + \mathcal{O}(z^2), \\ G^*(z) = G^*(0) + G_1^* z + \mathcal{O}(z^2), \end{cases} \quad \text{as } z \rightarrow 0.$$

Hence

$$n^{1/3}t(n^{-1/3}z) = in^{1/3}G(0) - iG_1 z + \mathcal{O}(n^{-1/3}),$$

as $n \rightarrow \infty$, which follows from (6.46). Using this, (6.14), and the fact that $G_1 \rightarrow G_1^*$ as $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} n^{1/3}t(n^{-1/3}z) = -2 \lim_{n \rightarrow \infty} n^{1/3}\gamma^{-1/2} \left(\frac{3}{2}\gamma^2 - \frac{1}{4\gamma} - \frac{5}{4}\tau^{4/3}\gamma \right) - iG_1^* z.$$

By inserting the limiting behavior for γ and τ as given in (6.2) and (2.25), we obtain the statement.

The proof for $s(z)$ is easier. Using (6.45) we obtain

$$n^{2/3}s(n^{-1/3}z) = \frac{1}{2}n^{2/3}e^{\frac{-3\pi i}{4}}F(0).$$

Using (6.13) we get

$$\lim_{n \rightarrow \infty} n^{2/3}s(n^{-1/3}z) = 2 \lim_{n \rightarrow \infty} n^{2/3}\gamma^{1/4} \left(-2\gamma^2 + \frac{1}{\gamma} + \tau^{4/3}\gamma \right),$$

which in combination with (6.2) and (2.25), finishes the proof. \square

From Lemma 6.7 it follows that (6.41) is well-defined (postponing the definition of E_n for a moment). Indeed, it follows from standard arguments that if the solution to RH problem 2.7 exists, it depends analytically on the parameters r_j , s , and t . Combining this observation with Lemma 6.7 we see that for the choice of r_j , s and t we made, the solution to RH problem 2.7 exists and hence (6.41) is well-defined for sufficiently large n .

Lemma 6.8. *Given definitions (6.43)–(6.46), the following formulas hold modulo $2\pi i$.*

$$\begin{cases} n(\tilde{\lambda}_1(z) + zK(-z)) = \psi_1(n^{2/3}z^{1/2}; r_1(z), n^{2/3}s(z)) + nt(z)z^{1/2}, \\ n(\tilde{\lambda}_2(z) + zK(-z)) = \psi_2(n^{2/3}z^{1/2}; r_2(z), n^{2/3}s(z)) - nt(z)z^{1/2}, \\ n(\tilde{\lambda}_3(z) + zK(-z)) = -\psi_2(n^{2/3}z^{1/2}; r_2(z), n^{2/3}s(z)) - nt(z)z^{1/2}, \\ n(\tilde{\lambda}_4(z) + zK(-z)) = -\psi_1(n^{2/3}z^{1/2}; r_1(z), n^{2/3}s(z)) + nt(z)z^{1/2}. \end{cases} \quad (6.49)$$

Proof. This is a straightforward verification. \square

It remains to define the prefactor

$$E_n(z) = e^{nzK(-z)} P^{(\infty)}(z) \tilde{A}^{-1} \times \text{diag} \left(n^{1/6} f(z)^{1/4}, n^{1/6} (-f(z))^{1/4}, n^{-1/6} (-f(z))^{-1/4}, n^{-1/6} f(z)^{-1/4} \right) \quad (6.50)$$

with \tilde{A} in (6.34). This prefactor is not analytic in the full disk $D(0, \rho n^{-1/3})$, but has a jump as indicated in the following lemma.

Lemma 6.9. *$E_n(z)$ is analytic in $D(0, \rho n^{-1/3}) \setminus (-\rho n^{-1/3}, 0]$ with a jump*

$$E_{n,+}(x) = E_{n,-}(x) \text{diag} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right), \quad -\rho n^{-1/3} < x < 0. \quad (6.51)$$

Proof. It follows from the definition of $E_n(z)$ and the fact that $P^{(\infty)}(z)$ is analytic for $z \in D(0, \rho n^{-1/3}) \setminus (-\rho n^{-1/3}, \rho n^{-1/3})$ that $E_n(z)$ is also analytic for $z \in D(0, \rho n^{-1/3}) \setminus \mathbb{R}$. For $x \in (0, \rho n^{-1/3})$, one checks that the jumps of $P^{(\infty)}$ and of the rightmost factor in (6.50) cancel each other out so that $E(x)$ is analytic for $x \in (0, \rho n^{-1/3})$. A similar calculation yields the jump (6.51) of $E_n(x)$ for $x \in (-\rho n^{-1/3}, 0)$. \square

Now let us check that $P^{(0)}(z)$ in (6.41) has the correct jumps. This is straightforward for the jumps on the 5 rays which are not \mathbb{R}^- . To check the jump for $x \in \mathbb{R}^-$, we calculate (see also [16])

$$\begin{aligned} P_+^{(0)}(x) &= E_{n,-}(x) \text{diag} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \\ &\quad \times N(n^{2/3} f_+(x); r_{1,+}(x), r_{2,+}(x), n^{2/3} s(x), n^{1/3} t(x)) \Lambda_+(x) \\ &= E_{n,-}(x) N(-n^{2/3} f_+(x); r_{2,+}(x), r_{1,+}(x), n^{2/3} s(x), n^{1/3} t(x)) \\ &\quad \times \Lambda_-(x) \text{diag} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \\ &= P_-^{(0)}(x) \text{diag} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right), \end{aligned}$$

where in the first step we used (6.41) and (6.51), and in the second step we used (6.37) and commuted with $\Lambda(x)$. The final equality follows from (6.47) and (6.41). This yields the required jumps on \mathbb{R}^- .

The following lemma states how the local parametrix matches with the outer parametrix on the boundary of the shrinking disk $D(0, \rho n^{-1/3})$, settling item (3) in the local RH problem 6.6.

Lemma 6.10. *For z on the shrinking circle the following matching condition uniformly holds as $n \rightarrow \infty$*

$$P^{(0)}(z) \left(P^{(\infty)}(z) \right)^{-1} = I + Z(z) + \mathcal{O}(n^{-1/12}), \quad |z| = \rho n^{-1/3}, \quad (6.52)$$

where

$$Z(z) = \frac{\hat{P}_\pm \tilde{N}_{1,\pm}(r_1(z), r_2(z), n^{2/3} s(z), n^{1/3} t(z)) \hat{Q}_\pm}{n^{1/3} z}, \quad \pm \text{Im } z > 0, \quad (6.53)$$

and where $\tilde{N}_{1,\pm}$, \hat{P}_\pm and \hat{Q}_\pm are defined in (6.36) and (6.24), respectively.

Proof. From (6.41), it follows that

$$P^{(0)}(z) \left(P^{(\infty)}(z) \right)^{-1} = E_n(z) N(n^{2/3} f(z); r_1(z), r_2(z), n^{2/3} s(z), n^{1/3} t(z)) \Lambda(z) \left(P^{(\infty)}(z) \right)^{-1}.$$

If $|z| = \rho n^{-1/3}$ then $|n^{2/3} z^{1/2}| \rightarrow \infty$ as $n \rightarrow \infty$. Hence, we can use (6.35), (6.50) and Lemma 6.8 to obtain, for $\pm \text{Im } z > 0$,

$$\begin{aligned} & P^{(0)}(z) \left(P^{(\infty)}(z) \right)^{-1} \\ &= P^{(\infty)}(z) \left(I + \frac{\tilde{N}_{1,\pm}(r_1(z), r_2(z), n^{2/3}s(z), n^{1/3}t(z))}{n^{1/3}z^{1/4}} + \mathcal{O}\left(\frac{1}{n^{2/3}z^{1/2}}\right) \right) \left(P^{(\infty)}(z) \right)^{-1}. \end{aligned}$$

This, together with (6.24), leads us to (6.52) and (6.53). \square

Note that $Z(z)$, as defined in (6.53), remains bounded as $n \rightarrow \infty$ with $|z| = \rho n^{-1/3}$.

6.6 Transformation $S \mapsto \tilde{R}$

Using the global parametrix $S^{(\infty)}(z)$ and the local parametrices $S^{(c)}$ and $S^{(0)}$ around c and 0 we define the transformation $S \mapsto \tilde{R}$ as

$$\tilde{R} = \begin{cases} S(S^{(0)})^{-1}, & \text{in the disk } D(0, \rho n^{-1/3}) \text{ around } 0, \\ S(S^{(c)})^{-1}, & \text{in a fixed disk around } c, \\ S(S^{(\infty)})^{-1}, & \text{elsewhere.} \end{cases} \quad (6.54)$$

Then \tilde{R} is defined and analytic outside Σ_S and the two disks around 0 and c , with an analytic continuation across those parts of Σ_S on which the jumps of the parametrices coincide with those of S . What remains are the jumps on a contour $\Sigma_{\tilde{R}}$ that consists of the two circles, the part of the interval $[c^*, \infty)$ outside the disk and the lips of local and global lenses outside the disks.

By setting the orientations of circles to be clockwise, we have that \tilde{R} satisfies the following RH problem.

RH problem 6.11. *The matrix-valued function \tilde{R} satisfies the following RH problem.*

(1) \tilde{R} is analytic in $\mathbb{C} \setminus \Sigma_{\tilde{R}}$.

(2) \tilde{R} satisfies the jump relation $\tilde{R}_+ = \tilde{R}_- J_{\tilde{R}}$ on $\Sigma_{\tilde{R}}$ with jump matrices

$$J_{\tilde{R}} = \begin{cases} S^{(0)}(S^{(\infty)})^{-1} & \text{on the boundary of } D(0, \rho n^{-1/3}), \\ S^{(c)}(S^{(\infty)})^{-1} & \text{on the boundary of the disk around } c, \\ S^{(\infty)} J_S (S^{(\infty)})^{-1} & \text{elsewhere on } \Sigma_{\tilde{R}}. \end{cases}$$

(3) As $z \rightarrow \infty$, we have

$$\tilde{R}(z) = I + \mathcal{O}(1/z).$$

The jump matrix $J_{\tilde{R}}$ is not close to the identity matrix on the shrinking circle around 0 , since by Lemma 6.10 we have

$$J_{\tilde{R}}(z) = S^{(0)}(z)(S^{(\infty)}(z))^{-1} = P^{(0)}(-z)(P^{(\infty)}(-z))^{-1} = I + Z(-z) + \mathcal{O}(n^{-1/12}), \quad \text{as } n \rightarrow \infty$$

uniformly for $|z| = \rho n^{-1/3}$, with $Z(z) = \mathcal{O}(1)$.

The other jump matrices, however, are close to the identity matrix as n gets large.

6.7 Final transformation $\tilde{R} \mapsto R$: nilpotent structure.

The presence of the bounded term $Z(z)$ in Lemma 6.10 requires an extra transformation.

Lemma 6.12. *The function $Z(z)$, defined in (6.53), has the following properties.*

- (a) $Z(z)$ is meromorphic in a neighborhood of zero with a simple pole in zero. Hence we can write

$$Z(z) = \frac{Z_0}{z} + \left(Z(z) - \frac{Z_0}{z} \right),$$

where $Z_0 = \text{Res}(Z, 0)$ is independent of z , and $Z(z) - Z_0/z$ is analytic in z .

- (b) $Z(z)$ is nilpotent of degree two, moreover

$$Z(z_1)Z(z_2) = 0, \quad \text{for any } z_1, z_2 \text{ in a neighborhood of the origin.}$$

Proof. To prove (a) note that it is clear from (6.53) that $Z(z)$ is analytic in a neighborhood of zero with cut along the real line. It is then sufficient to show that there is no jump on the real line. When $x \in (0, \rho)$, there are no jumps for the functions r_1, r_2, s , and t . Hence, it follows from (6.36) and (6.34) that

$$\tilde{N}_{1,+} = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & -i & -1 & 0 \\ 0 & 1 & -i & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} \tilde{N}_{1,-} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & -i & 1 & 0 \\ 0 & -1 & -i & 0 \\ i & 0 & 0 & 1 \end{pmatrix}. \quad (6.55)$$

On account of (6.26) and (6.29), we obtain from (6.53) that

$$Z_+(x) = \frac{1}{4n^{1/3}x} \hat{P}_- \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} \tilde{N}_{1,-} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix} \hat{Q}_- = Z_-(x),$$

where the second equality follows from (6.28) and (6.31). On the negative real line we also need (6.38). Indeed, for $-\rho < x < 0$ and $\zeta_{\pm} := n^{2/3}x_{\pm}^{1/2}$, we have

$$\begin{aligned} Z_+(x) &= \frac{\hat{P}_+}{x_+^{3/8}} \frac{\tilde{N}_{1,+}(r_{1,+}(x), r_{2,+}(x), n^{2/3}s(x), n^{1/3}t(x))}{n^{1/3}x_+^{1/4}} \frac{\hat{Q}_+}{x_+^{3/8}} \\ &= \frac{\hat{P}_-}{x_-^{3/8}} \frac{\tilde{N}_{1,-}(r_{2,+}(x), r_{1,+}(x), n^{2/3}s(x), n^{1/3}t(x))}{(-\zeta_+)^{1/2}} \frac{\hat{Q}_-}{x_-^{3/8}}, \end{aligned}$$

where we have used (6.38) and the relations (6.27) and (6.30). Combining this with (6.47) and the fact that $\zeta_+ = -\zeta_-$ we see that $Z(z)$ is continuous across $(-\rho, 0)$ and, thus, also analytic in a punctured neighborhood of the origin. Recalling (6.53) we then obtain (a).

(b) follows from (6.53) and the observation that $\hat{Q}_{\pm}\hat{P}_{\pm} = 0$, see (6.25). \square

As a corollary we also get

$$Z(z)Z_0 = 0, \quad Z_0Z(z) = 0, \quad Z_0^2 = 0. \quad (6.56)$$

Then we define the transformation $\tilde{R} \mapsto R$ as

$$R(z) = \begin{cases} \tilde{R}(z) \left(I + Z(-z) + \frac{Z_0}{z} \right), & \text{for } z \in D(0, \rho n^{-1/3}) \setminus \Sigma_{\tilde{R}}, \\ \tilde{R}(z) \left(I + \frac{Z_0}{z} \right), & \text{for } z \in \mathbb{C} \setminus (D(0, \rho n^{-1/3}) \cup \Sigma_{\tilde{R}}). \end{cases} \quad (6.57)$$

Then R is defined and analytic in $\mathbb{C} \setminus \Sigma_R$ where $\Sigma_R = \Sigma_{\tilde{R}}$ and R satisfies a RH problem of the following form.

RH problem 6.13.

- (1) R is analytic in $\mathbb{C} \setminus \Sigma_R$.
- (2) R satisfies the jump conditions $R_+ = R_- J_R$ on Σ_R , with J_R described below.
- (3) $R(z) = I + \mathcal{O}(1/z)$ as $z \rightarrow \infty$.

The jump matrix J_R for $|z| = \rho n^{-1/3}$ is by (6.57) and (6.56)

$$\begin{aligned} J_R(z) &= \left(I + Z(-z) + \frac{Z_0}{z} \right)^{-1} J_{\tilde{R}}(z) \left(I + \frac{Z_0}{z} \right) \\ &= \left(I - Z(-z) - \frac{Z_0}{z} \right) \left(I + Z(-z) + \mathcal{O}(n^{-1/12}) \right) \left(I + \frac{Z_0}{z} \right) = I + \mathcal{O}(n^{-1/12}), \end{aligned}$$

where we also use the fact that $Z(z)$ and Z_0/z are bounded for $|z| = \rho n^{-1/3}$.

The transformation (6.57) does not change the jump matrices on the other parts of Σ_R in an essential way. Hence J_R tends to the identity matrix on these parts as well, with a rate of convergence that is the same as that for J_V .

We have now achieved the goal of the steepest descent analysis. $R(z)$ tends to the identity matrix as $z \rightarrow \infty$ and the jump matrices for R tend to the identity matrix as $n \rightarrow \infty$, both uniformly on Σ_R and in $L^2(\Sigma_R)$. By standard arguments, see [15] and in particular [9] for the case of a moving contour, this leads to the conclusion of our steepest descent analysis

$$R(z) = I + \mathcal{O}\left(\frac{1}{n^{1/12}(1+|z|)}\right), \quad (6.58)$$

as $n \rightarrow \infty$, uniformly for $z \in \mathbb{C} \setminus \Sigma_R$.

6.8 Proof of Theorem 2.8

The idea of the proof is similar to that of the proof of Theorem 2.6, i.e. we write the expression for the correlation kernel in terms of R instead of Y by unfolding all transformations performed in the steepest descent analysis. Since the first few transformations are the same as in the regular cases, we see from the arguments in Section 5.8 that if $0 < x, y < c$,

$$\begin{aligned} e^{n(y-x)} K_n(x, y) &= \frac{y^{\nu/2} x^{-\nu/2}}{2\pi i(x-y)} \begin{pmatrix} -e^{n\lambda_{1,+}(y)} & e^{n\lambda_{2,+}(y)} & 0 & 0 \end{pmatrix} \\ &\quad \times S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-n\lambda_{1,+}(x)} & e^{-n\lambda_{2,+}(x)} & 0 & 0 \end{pmatrix}^T. \end{aligned} \quad (6.59)$$

Moreover, for $x, y \in D(0, \rho n^{-1/3})$, we have by (6.54) and (5.60)

$$S_+(z) = \tilde{R}_+(z) S_+^{(0)}(z) = \tilde{R}_+(z) P_-^{(0)}(-z) J \operatorname{diag}(e^{\nu\pi i/2}, e^{-\nu\pi i/2}, e^{\nu\pi i/2}, e^{-\nu\pi i/2}), \quad z = x, y,$$

where J is given in (5.61). It then follows that

$$\begin{aligned} e^{n(y-x)} K_n(x, y) &= \frac{y^{\nu/2} x^{-\nu/2}}{2\pi i(x-y)} \begin{pmatrix} 0 & 0 & e^{n\lambda_{2,+}(y)+\nu\pi i/2} & -e^{n\lambda_{1,+}(y)-\nu\pi i/2} \end{pmatrix} \\ &\quad \times \left(P_-^{(0)}(-y) \right)^{-1} \tilde{R}^{-1}(y) \tilde{R}(x) P_-^{(0)}(-x) \begin{pmatrix} 0 & 0 & e^{-n\lambda_{2,+}(x)-\nu\pi i/2} & e^{-n\lambda_{1,+}(x)+\nu\pi i/2} \end{pmatrix}^T. \end{aligned}$$

By (6.41), (6.42), and (6.22) this becomes

$$\begin{aligned}
e^{n(y-x)} K_n(x, y) &= \frac{y^{\nu/2} x^{-\nu/2}}{2\pi i(x-y)} \begin{pmatrix} 0 & 0 & e^{\nu\pi i/2} & -e^{-\nu\pi i/2} \end{pmatrix} \\
&\times N_- \left(n^{2/3}(-y)^{1/2}; r_{1,-}(-y), r_{2,-}(-y), n^{2/3}s(-y), n^{1/3}t(-y) \right)^{-1} E_{n,-}^{-1}(-y) \tilde{R}^{-1}(y) \tilde{R}(x) E_{n,-}(-x) \\
&\times N_- \left(n^{2/3}(-x)^{1/2}; r_{1,-}(-x), r_{2,-}(-x), n^{2/3}s(-x), n^{1/3}t(-x) \right) \begin{pmatrix} 0 & 0 & e^{-\nu\pi i/2} & e^{\nu\pi i/2} \end{pmatrix}^T.
\end{aligned} \tag{6.60}$$

Now we scale x and y with n such that

$$x = \frac{u}{n^{4/3}} \quad \text{and} \quad y = \frac{v}{n^{4/3}}, \tag{6.61}$$

where $u, v > 0$. Then for large n , x and y belong to the disk $D(0, \rho n^{-1/3})$, so that (6.60) holds. We want to take the limit as $n \rightarrow \infty$. Note that under these conditions

$$\lim_{n \rightarrow \infty} e^{n(y-x)} = 1,$$

and by (6.48)

$$\begin{aligned}
r_j(z) &\rightarrow 2, & j &= 1, 2, \\
n^{2/3}s(z) &\rightarrow \frac{1}{2}(a^2 - 5b), \\
n^{1/3}t(z) &\rightarrow 2a,
\end{aligned}$$

as $n \rightarrow \infty$ and $z = x, y$. Furthermore, it follows from (6.58) and Cauchy's formula that

$$R^{-1}(y)(R(y) - R(x)) = \mathcal{O}\left(\frac{x-y}{n^{1/12}}\right) = \mathcal{O}\left(n^{-17/12}\right), \tag{6.62}$$

as $n \rightarrow \infty$ where the constant is uniform for u, v in compact subsets of \mathbb{R} . Then also

$$\begin{aligned}
\tilde{R}^{-1}(y) \tilde{R}(x) &= \left(I + Z(-y) + \frac{Z_0}{y} \right) R^{-1}(y) R(x) \left(I + Z(-x) + \frac{Z_0}{x} \right)^{-1} \\
&= \left(I + Z(-y) + \frac{Z_0}{y} \right) R^{-1}(y) R(x) \left(I - Z(-x) - \frac{Z_0}{x} \right) = I + \mathcal{O}(n^{-1}),
\end{aligned} \tag{6.63}$$

as $n \rightarrow \infty$, where the constant is again uniform for u, v in compact subsets of \mathbb{R} . Here we used (6.56) to invert the rightmost matrix. To prove the last equality in (6.63), note that the matrix function $Z(z) - \frac{Z_0}{z}$, which is analytic by Lemma 6.12(a), can be written as a power series in the variable $n^{1/3}z$ with coefficients having a limit for $n \rightarrow \infty$, thanks to (6.53) and (6.48). Applying this with $z = x, y$ given in (6.61) we get the claimed $\mathcal{O}(n^{-1})$ estimate in (6.63). In fact, the same reasoning yields the following more precise version of (6.63),

$$\tilde{R}^{-1}(y) \tilde{R}(x) = I + \hat{P}_+ \mathcal{O}(n^{-1}) + \mathcal{O}(n^{-1}) \hat{Q}_+ + o(n^{-1}), \quad n \rightarrow \infty, \tag{6.64}$$

where the matrices \hat{P}_+ , \hat{Q}_+ originate from (6.53). (We could also write \hat{P}_- , \hat{Q}_- instead.)

Next we estimate the factor $E_n(z)$ given in (6.50). We claim that the transformed matrix

$$\tilde{E}_n(z) := E_n(z) \operatorname{diag} \left(\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right) \operatorname{diag}(\zeta^{-1/4}, \zeta^{1/4}, \zeta^{-1/4}, \zeta^{1/4}), \quad \zeta = n^{4/3}z, \tag{6.65}$$

is analytic near $z = 0$. Indeed it has no jumps, by virtue of Lemma 6.9, and moreover it behaves as $\mathcal{O}(z^{-3/4})$ as $z \rightarrow 0$ so there is no pole at $z = 0$.

Lemma 6.14. *We have*

$$\lim_{n \rightarrow \infty} \tilde{E}_n^{-1}(-y) \tilde{R}^{-1}(y) \tilde{R}(x) \tilde{E}_n(-x) = I, \quad (6.66)$$

uniformly for u, v in compact subsets of \mathbb{R} .

Proof. In the proof below all the \mathcal{O} and o terms will be uniform for u, v in compact subsets of \mathbb{R} . We start by writing

$$\begin{aligned} \tilde{E}_n^{-1}(-y) \tilde{R}^{-1}(y) \tilde{R}(x) \tilde{E}_n(-x) &= \tilde{E}_n^{-1}(-y) \tilde{E}_n(-x) \\ &\quad + \tilde{E}_n^{-1}(-y) (\tilde{R}^{-1}(y) \tilde{R}(x) - I) \tilde{E}_n(-x). \end{aligned} \quad (6.67)$$

Let us estimate the first term in the right hand side. We have $\tilde{E}_n(z)^{\pm 1} = \mathcal{O}(n^{1/2})$, $z = x, y$, as $n \rightarrow \infty$, which is a special case of (6.69) below. Then by the analyticity of $\tilde{E}_n(z)$ we obtain

$$\tilde{E}_n^{-1}(y) (\tilde{E}_n(y) - \tilde{E}_n(x)) = \mathcal{O}((x - y)n) = \mathcal{O}(n^{-1/3}), \quad (6.68)$$

as $n \rightarrow \infty$. Consequently the first term in the right hand side of (6.67) goes to the identity matrix for $n \rightarrow \infty$.

Next we estimate the second term in the right hand side of (6.67). On account of (6.61), (6.50) and (6.24) we have

$$\tilde{E}_n(z) = \hat{P}_+ \mathcal{O}(n^{1/2}) + o(n^{1/2}), \quad \tilde{E}_n(z)^{-1} = \mathcal{O}(n^{1/2}) \hat{Q}_+ + o(n^{1/2}), \quad z = x, y, \quad (6.69)$$

as $n \rightarrow \infty$. From (6.69) and (6.64) we see that

$$\begin{aligned} \tilde{E}_n^{-1}(-y) (\tilde{R}^{-1}(y) \tilde{R}(x) - I) \tilde{E}_n(-x) \\ = \mathcal{O}(n^{1/2}) \left(\hat{Q}_+ \hat{P}_+ \mathcal{O}(n^{-1}) \hat{P}_+ + \hat{Q}_+ \mathcal{O}(n^{-1}) \hat{Q}_+ \hat{P}_+ \right) \mathcal{O}(n^{1/2}) + o(1) = o(1), \end{aligned}$$

for $n \rightarrow \infty$, where the second equality follows from the orthogonality relation $\hat{Q}_+ \hat{P}_+ = 0$ in (6.25). Hence the second term in the right hand side of (6.67) goes to zero for $n \rightarrow \infty$. \square

To use the above lemma, we should first express the matrix E_n in (6.60) in terms of its transformed counterpart \tilde{E}_n in (6.65). This substitution releases an extra factor which multiplies from the left the matrix N in (6.60). By combining this with the above estimates we find

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^{4/3}} K_n \left(\frac{u}{n^{4/3}}, \frac{v}{n^{4/3}} \right) &= \frac{u^{-\nu/2} v^{\nu/2}}{2\pi i(u - v)} \begin{pmatrix} 0 & 0 & e^{\nu\pi i/2} & -e^{-\nu\pi i/2} \end{pmatrix} \\ \times \hat{N} \left(-iv^{1/2}; 2, 2, \frac{1}{2}(a^2 - 5b), 2a \right)^{-1} \hat{N} \left(-iu^{1/2}; 2, 2, \frac{1}{2}(a^2 - 5b), 2a \right) &\begin{pmatrix} 0 & 0 & e^{-\nu\pi i/2} & e^{\nu\pi i/2} \end{pmatrix}^T, \end{aligned} \quad (6.70)$$

with

$$\hat{N} \left(-i\zeta^{1/2} \right) := \text{diag}(\zeta^{1/4}, \zeta^{-1/4}, \zeta^{1/4}, \zeta^{-1/4}) \text{diag} \left(\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \right) N \left(-i\zeta^{1/2} \right),$$

for $\zeta = u, v$. Equivalently, by (6.32), (6.37) and (2.30),

$$\hat{N} \left(-i\zeta^{1/2} \right) = \text{diag}(1, -i, i, 1) \widehat{M}(\zeta) \text{diag} \left(1, i, \begin{pmatrix} 0 & -1 \\ -i & 0 \end{pmatrix} \right).$$

Inserting this in (6.70) we get

$$\lim_{n \rightarrow \infty} \frac{1}{n^{4/3}} K_n \left(\frac{u}{n^{4/3}}, \frac{v}{n^{4/3}} \right) = \frac{u^{-\nu/2} v^{\nu/2}}{2\pi i(u-v)} \begin{pmatrix} 0 & 0 & e^{-\nu\pi i/2} & ie^{\nu\pi i/2} \end{pmatrix} \\ \times \widehat{M} \left(v; \frac{1}{2}(a^2 - 5b), 2a, \nu \right)^{-1} \widehat{M} \left(u; \frac{1}{2}(a^2 - 5b), 2a, \nu \right) \begin{pmatrix} 0 & 0 & -e^{\nu\pi i/2} & -ie^{-\nu\pi i/2} \end{pmatrix}^T,$$

where we recall that $\widehat{M}(z) = \widehat{M}(z; s, t, \nu)$ is defined in (2.30) with $M(\zeta) = M(\zeta; s, t, \nu)$ denoting the unique solution to RH problem 2.7 with parameters given in (2.32). The above kernel is clearly equal to the right hand side of (2.33). This completes the proof of Theorem 2.8.

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